A Time-Frequency Distribution Concentrated Along the Instantaneous Frequency

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Abstract— **A time-frequency distribution that produces high concentration at the instantaneous frequency for an arbitrary signal is proposed. This distribution may be treated as a variant of the L-Wigner distribution, but it also satisfies unbiased energy condition, time marginal, as well as the frequency marginal in the case of asymptotic signals. The presented theory is illustrated by examples.**

I. Introduction

TIME–FREQUENCY analysis has attracted attention of many researchers. The main challenge in this area lies in the fact that many fundamental questions are still waiting for viable answers. The whole variety of tools for time–frequency analysis, mainly rendered in the form of energy distributions in the time–frequency plane, has been proposed (for a complete list, see the review papers [1] and [2] and references therein). Cohen has shown that all shift-covariant quadratic time–frequency distributions are simply special cases of a general class of distributions obtained for a particular choice of an arbitrary function (kernel). Out of the Cohen class, the Wigner distribution (WD) is the only one (with signal independent kernel) that produces the ideal concentration along instantaneous frequency $\phi'(t)$ ($\mathrm{WD}\,(t,\omega) = 2\pi A^2\delta(\omega-\phi'(t))$ for the linear frequency modulated signals $x(t) = A\exp(\phi(t))$ and $\phi(t) = bt^2/2 + ct + d$ [3], [4], [5], [18], [19]. In order to improve the concentration of *monocomponent signals*, when the instantaneous frequency is polynomial function of time, the polynomial WD is proposed [6], [7]. A similar idea for improving the distribution concentration of the signal whose phase is polynomial up to the fourth order was presented in [8]. In order to improve distribution concentration for a signal with an arbitrary nonlinear instantaneous frequency, the L-Wigner distribution (LWD) was proposed and studied in [8], [9], [10], [4], and [5]. The polynomial WD, as well as the LWD, are closely related to the time-varying higher odder spectra [7], [9], [10], [11]. They do not preserve the usual marginal properties [1], [2], but they do satisfy the generalized forms of the marginals. For example, the time marginal in the LWD is the generalized power $|x(t)|^{2L}$, rather than $|x(t)|^2$. Here, we will present a variant of the LWD obtained by scaling the phase and τ axis by an integer L while keeping the signals' amplitudes unchanged. This distribution may achieve high concentration at the instantaneous frequency—as high as the LWD—while at the same time satisfying time marginal and for asymptotic signals frequency marginal.

Manuscript received June 19, 1995. The associate editor coordinating the review of this paper and approving it for publication was Prof. J. M. F. Moura.

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Publisher Item Identifier S 1070-9908(96)02748-4.

II. Definition and Properties

The scaled variant of the L-Wigner distribution (SD) of a signal $x(t)$, in its pseudo form, is defined by

$$\mathrm{SD}_L\,(t,\omega) = \int_{-\infty}^{\infty} w_L(\tau)x^{[L]}\Big(t+\frac{\tau}{2L}\Big)x^{[L]*} \cdot \Big(t-\frac{\tau}{2L}\Big)e^{-j\omega\tau}\,d\tau \qquad (1)$$

where $x^{[L]}(t)$ is the modification of $x(t)$ obtained by multiplying the phase function by L while keeping the amplitude unchanged:

$$x^{[L]}(t) = A(t)e^{jL\phi(t)}. \qquad (2)$$

The word "pseudo" will be used to indicate the presence of window $w_L(\tau)$. For $L = 1$, the WD follows.[1]

The distribution defined by (1) satisfies the time marginal and unbiased energy condition for any L

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{SD}_L\,(t,\omega)\,d\omega = A^2(t) \quad \text{and}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{SD}_L\,(t,\omega)\,d\omega\,dt = \int_{-\infty}^{\infty}A^2(t)\,dt = E_x \qquad (3)$$

where $w_L(0) = 1$ is assumed.

The frequency marginal is satisfied for asymptotic signals, as well. Substituting $\tau/L \to \tau$ in (1), we get

$$\int_{-\infty}^{\infty}\mathrm{SD}_L\,(t,\omega)\,dt = L\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A\Big(t+\frac{\tau}{2}\Big)A\Big(t-\frac{\tau}{2}\Big) \cdot e^{jL[\phi(t+(\tau/2))-\phi(t-(\tau/2))-\omega\tau]}\,d\tau\,dt = L|X_n(L\omega)|^2$$

[1]The original idea for this distribution stems from the very well-known quantum mechanics forms. For a signal $x(t) = A(t)e^{j\phi(t)}$, we form a function $\psi(\lambda) = A(\lambda)e^{jL\phi(\lambda)}$ that corresponds (with $L = 1/\hbar$) to the Wentzel solution of the Schroedinger's equation or to the Feynman's path integral. This form applied to the original quantum mechanics form of the Wigner distribution $\mathrm{WD}\,(\lambda,p) = \int \psi(\lambda+\hbar\tau/2)\psi^*(\lambda-\hbar\tau/2)e^{-jp\tau}\,d\tau$ produces the SD exactly. Of course, in signal processing, we are not restricted to the real word value of $\hbar \sim 10^{-34}$, nor would this value be appropriate for applications. It will be shown that with L slightly greater than 1 ($L = 2, 4, \cdots$), we may significantly benefit with respect to the distribution concentration (uncertainty is of order $1/L^2$) while at the same time keeping other important properties of the time–frequency presentation invariant.

According to the stationary phase method [12]

$$X_n(L\omega) = \int_{-\infty}^{\infty} A(t)e^{jL\phi(t)-jL\omega t}\,dt$$
$$= A(t_0)e^{jL\phi(t_0)-jL\omega t_0}\sqrt{\frac{2\pi j}{L\phi''(t_0).}} \qquad (4)$$

This holds for $L \to \infty$ for any signal with continuous $A(t)$ and for asymptotic signals [12], [13] (signals with $|A'(t)| \ll |\phi'(t)|$) for any L, including $L = 1$ (t_0 is the solution of $\phi'(t_0) - \omega = 0$). It is very easy to conclude from (4) that for asymptotic signals

$$L|X_n(L\omega)|^2 = |X(\omega)|^2$$

i.e., the SD, with $w_L(\tau) = 1$, satisfies the frequency marginal in this case as well.

Theorem: For any signal $x(t) = A(t)e^{j\phi(t)}$ having finite derivatives of the phase function $\phi(t)$ and continuous amplitud $A(t)$, the SD for $L \to \infty$ is concentrated along the instantaneous frequency

$$\lim_{L\to\infty} \mathrm{SD}_L(t,\omega) = A^2(t)W(\omega - \phi'(t)) \qquad (5)$$

where $w(\tau)$ is a finite duration window $W(\omega) = \mathrm{FT}\{w(\tau)\}$.

Proof: For a signal of the form $x(t) = A(t)e^{j\phi(t)}$, expanding $\phi(t \pm \tau/2L)$ into a Taylor series around t up to the third order term, we get

$$\mathrm{SD}_L(t,\omega) = \int_{-\infty}^{\infty} w(\tau)A\Big(t+\frac{\tau}{2L}\Big)A\Big(t-\frac{\tau}{2L}\Big)$$
$$\cdot\, e^{j\phi'(t)\tau}e^{j(\phi^{(3)}(t+\tau_1)+\phi^{(3)}(t-\tau_2)/3!L^2)(\tau/2)^3}$$
$$\cdot\, e^{-j\omega\tau}\,d\tau \qquad (6)$$

where τ_1, τ_2 are variables $0 \le |\tau_{1,2}| \le |\tau/2L|$. If $\phi^{(3)}(\tau)$ and $\phi^{(n)}(\tau), n > 3$ are finite, then for a large L and finite duraton $w(\tau)$, the value $\lim_{L\to\infty}[w(\tau)\exp(j(\phi^{(3)}(t+\tau_1) + \phi^{(3)}(t-\tau_2)/3!L^2)(\tau^3/8))] = w(\tau)$. In addition, for continuous $A(t)$, $A(t+(\tau/2L))A(t-(\tau/2L)) \to A^2(t)$ holds, and the form stated in the theorem follows. **Q.E.D.**

Further properties of the SD that are invariant with respect to L (time-shift, modulation, time-support, frequency support for asymptotic signals, $\cdots$) may be easily derived (proved), following the ones for the Cohen class of distribution [1], [2] or the ones for the L-class of distributions [14]. Form (1) may be applied to any other transform or distribution. For example, the modified version the short-time Fourier transform is

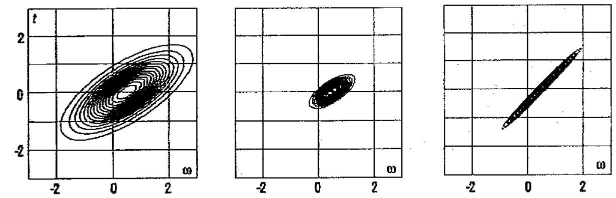
$$\mathrm{MST}_L(t,\omega) = \int_{-\infty}^{\infty} w_L(\tau)x^{[L]}(t+\tau/L)\exp(-j\omega\tau)\,d\tau.$$

The relation between $\mathrm{MST}_L(t,\omega)$ and $\mathrm{SD}_L(t,\omega)$ is $\mathrm{SD}_L(t,\omega) = (1/\pi)\int \mathrm{MST}_L(t,\omega+\theta)\mathrm{MST}^*_L(t,\omega-\theta)\,d\theta$.

III. Examples

Example 1: Consider the *Gaussian chirp signal* of the form

$$x(t) = Ae^{-at^2/2}e^{jbt^2/2+jct}. \qquad (7)$$



(a) (b) (c)

Fig. 1. Time–frequency representation of a Gaussian chirp signal: a) Wigner distribution, b) L-Wigner distribution, c) the SD distribution with $L = 8$.

a) The *L-Wigner distribution* has the form

$$\mathrm{LWD}_L(t,\omega) = \int_{-\infty}^{\infty} x^L\Big(t+\frac{\tau}{2L}\Big)x^{*L}\Big(t-\frac{\tau}{2L}\Big)e^{-j\omega\tau}\,d\tau$$
$$= A^{2L}e^{-aLt^2}\sqrt{\frac{4\pi L}{a}}e^{-((\omega-bt-c)^2/a/L)}.$$

For small $a/L \to 0$, we get

$$\mathrm{LWD}_L(t,\omega) = A^{2L}e^{-aLt^2}2\pi\delta(\omega - bt - c) \qquad (8)$$

This distribution, for large L, produces the ideal concentration at the instantaneous frequency for any a, but the amplitude is raised to the Lth power. The WD ($L = 1$) produces the complete concentration at the instantaneous frequency only for $a \to 0$, i.e., for the purely linear FM signal [3]–[5]. For any other a, the distribution is spread around the instantaneous frequency, In Fig. 1(a) and (b), ($A = a = b = 2c = 1$).

b) The SD of the Gaussian chirp signal is

$$\mathrm{SD}_L(t,\omega) = \int_{-\infty}^{\infty} x^{[L]}\Big(t+\frac{\tau}{2L}\Big)x^{[L]*}\Big(t-\frac{\tau}{2L}\Big)e^{-j\omega\tau}\,d\tau$$
$$= A^2e^{-at^2}\sqrt{\frac{4\pi}{a}}Le^{-((\omega-bt-c)^2/a/L^2)}.$$

For $a/L^2 \to 0$, we have the following:

$$\mathrm{SD}_L(t,\omega) = A^2e^{-at^2}2\pi\delta(\omega - bt - c). \qquad (9)$$

This is the ideal time–frequency concentration at the instantaneous frequency for any a. The convergence toward to the complete concentrated distribution is of order L^2; see Fig. 1(c).

Example 2: Consider a noisy signal whose amplitude and phase variations are of the same order:

$$x(t) = A(t)e^{j\phi(t)} + n(t)$$
$$= e^{j15\pi(t+0.8)^2} + e^{-j12\sin[3\pi/2(t+1)]-j30\pi t+j8\pi(t+0.5)^2}$$
$$+\, n(t) \qquad (10)$$

where $n(t)$ represents Gaussian white complex noise of variance $\sigma^2 = 1$. In this case, any atempt to obtain the instantaneous frequency directly would fail. The WD and LWD of signal (10) calculated using the Hanning window of the width $1/L$ and $N = 64$ samples inside the window are presented in Fig. 2(a) and (b). For the LWD realization, the recursive approach described in [4], [5], [9], [10], and [17] is used. The SD with $L = 2$, in Fig. 2(c) is calculated using the same recursive approach, starting from the modified STFT with $L = 2$, $\mathrm{MST}_2(t,\omega) = \mathrm{FT}[w_2(\tau)x^{[2]}(t+\tau/2)]$ and $\mathrm{SD}_2(t,\omega) =$

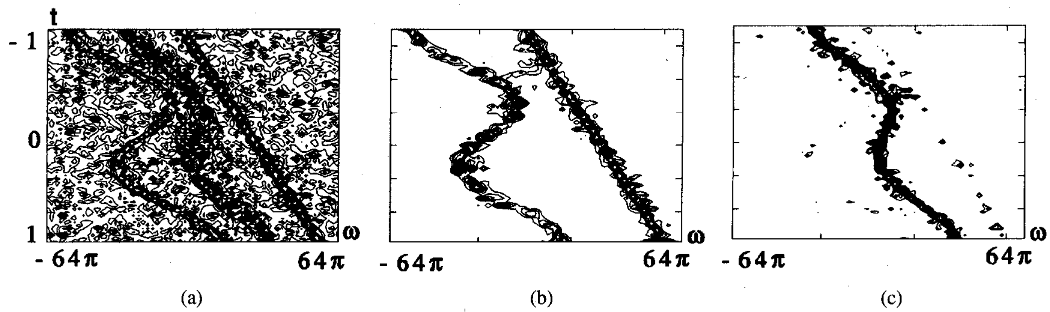


Fig. 2. Time–frequency representation of a signal with fast amplitude variations: a) Wigner distribution, b) L-Wigner distribution with $L = 2$, c) the SD distribution with $L = 2$.

$(1/\pi) \int P(\theta) \mathrm{MST}_2(t, \omega+\theta) \mathrm{MST}_2^*(t, \omega-\theta)\, d\theta$. For details on numerical realization of the last relation, see [4], [5], [9], and [17]. Note that the sampling interval in the $\mathrm{MST}_2(t, \omega)$ is the same as in the WD. The results presented in Fig. 2 are in complete agreement with the above discussion.

Note that modification (3) of the resulting phase in a multicomponent signal is not the same as the modifications of the individual components' phases. In the case of multicomponent signals, the resulting signal may be written in the form $x(t) = A(t) \exp(j\phi(t))$ so that the SD produces the complete concentration at the instantaneous frequency $\phi'(t)$ in the sense of the Theorem[2]; see Fig. 2(c). Frequency $\phi'(t)$, in the case of multicomponent signals, is the mean conditional frequency $\phi'(t) = \langle\omega\rangle_t = [\int \omega\, \mathrm{SD}(t, \omega) d\omega]/[\int \mathrm{SD}(t, \omega)\, d\omega]$, for any L, including the WD with $L = 1$.

Signal (10) may be treated in two ways: as a multicomponent one or as a FM signal whose amplitude varies rapidly. Depending on its true nature, we may, for its analysis, use the LWD or the SD; see Fig. 2(b) or (c).[3]

IV. Conclusion

The phase-scaled variant of the L-Wigner distribution is presented. This distribution may produce the ideal signal power concentration at the instantaneous frequency.

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[2] The L-Wigner distribution of the signal $x_M(t) = A^{1/L}(t) \exp(j\phi(t))$ is simply the SD. If the signal $x(t)$ is multicomponent [13], [15], [16], then the amplitude $A(t)$ variations are of the same order as the variations of $\phi(t)$. However, with a large L, the modified signal $x_M(t)$ assumes alomost constant unity amplitude, i.e., it becomes a monocomponent one with the instantaneous frequency $\phi'(t)$.

[3] In the meantime, since this paper was accepted for publication, we derived a method for the realization of the SD such that, in the case of multicomponent signals, it may be equal to the sum of the SD's of each component separately [20].