### REFERENCES

[1] L. Cohen, *Time-Frequency Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1995.

[2] R. G. Baraniuk and D. L. Jones, "A signal-dependent time-frequency representation: Optimal kernel design," *IEEE Trans. Signal Processing,* vol. 41, pp. 1589–1602, Apr. 1993.

[3] A. H. Costa and G. F. Boudreaux Bartels, "Design of time-frequency representations using multiform tiltable kernels," in *Proc. IEEE Int. Conf. TF/TS Anal.,* Philadelphia, PA, Oct. 25–28, 1994, pp. 205–208.

[4] M. G. Amin, G. T. Venkatesan, and J. F. Carroll, "A constrained weighted least squares approach for time-frequency distribution kernel design," *IEEE Trans. Signal Processing.,* vol. 44, pp. 1111–1120, May 1996.

[5] T. W. Parks and C. S. Burrus, *Digital Filter Design.* New York: Wiley, 1987.

[6] B. D. Van Veen and K. M. Buckley, "Beamforming: A versatile approach to spatial filtering," *IEEE Acoust., Speech, Signal Processing Mag.,* pp. 4–24, Apr. 1988.

[7] F. J. Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proc. IEEE,* vol. 66, pp. 51–83, Jan. 1978.

[8] J. Jeong and W. J. Williams, "Kernel design for reduced interference distributions," *IEEE Trans. Signal Processing,* vol. 40, pp. 402–412, Feb. 1992.

[9] C. Tseng and L. J. Griffiths, "A simple algorithm to achieve desired patterns for arbitrary arrays," *IEEE Trans. Signal Processing,* vol. 40, pp. 2737–2746, Nov. 1992.

[10] G. T. Venkatesan and M. G. Amin, "Time-frequency distribution kernels using FIR filter design techniques," in *Proc. Asilomar Conf. Circuits, Syst. Comput.,* Pacific Grove, CA, Nov. 1995.


# Further Results on the Minimum Variance Time-Frequency Distribution Kernels

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***Abstract*— Results for the minimum variance kernel, presented by Hearon and Amin, for the complex Gaussian white noise with independent real and imaginary parts remain valid for real noise and approximately valid for analytic noise. These results are extended to the real and analytic noisy signals cases.**

## I. INTRODUCTION

The influence of complex noise, with independent real and imaginary parts, on the Cohen class of distributions is analyzed in [1] by Hearon and Amin. Of the Cohen class of distributions satisfying marginal and time-support conditions, with respect to the noise influence, it has been shown that the Born–Jordan distribution is optimal. In this correspondence, we will extend the analysis from [1] to two very important types of noise: real and analytic. From this analysis, it will be shown that the results and conclusions obtained in [1] remain valid with respect to these two forms of noise. In the second part of the correspondence, Amin's recent results for the noisy signal case [2] are extended on the real and analytic noisy signal forms.

## II. NOISE ANALYSIS

The discrete time form of the Cohen class of distributions, for signal $x(n)$, is given by [1]–[3]

$$C_x(n,\omega;\varphi) = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\varphi(m,k) \times x(n+m+k)x^*(n+m-k)e^{-j2\omega k}. \quad (1)$$

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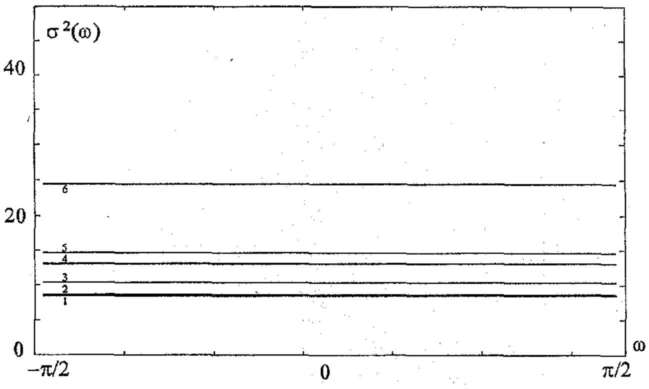


Fig. 1. Variance in the case of complex noise with independent real and imaginary parts for 1) Born–Jordan distribution, 2) optimal auto-term distribution, 3) Choi–Williams distribution, 4) Butterworth distribution, 5) sinc distribution, and 6) pseudo-Wigner distribution.

Let us suppose, as in [1], that $x(n)$ is a Gaussian noise with variance $\sigma_x^2$. The variance of the Cohen class of the distributions' estimator $\sigma^2(\omega) = \mathrm{var}[C_x(n,\omega;\varphi)]$ is given by [1], [2], [4]

$$\begin{aligned}\sigma^2(\omega) = &\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} \\ &\cdot \varphi(m_1,k_1)\varphi^*(m_2,k_2) \\ &\times [R_{xx}(m_1-m_2+k_1-k_2) \\ &\cdot R_{xx}^*(m_1-m_2-k_1+k_2) \\ &+ R_{xx^*}(m_1-m_2+k_1+k_2) \\ &\times R_{x^*x}(m_1-m_2-k_1-k_2)]e^{-j2\omega(k_1-k_2)} \end{aligned} \quad (2)$$

where $R_{xx}(m)$ is the autocorrelation function of $x(n)$.

### A. Complex Noise

Assume complex white Gaussian noise $x(n)$ with independent real and imaginary parts having equal variances $\sigma_x^2/2$. The variance of the Cohen class estimator has been derived in [1] in the form

$$\sigma^2(\omega) = \sigma_x^4 \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi(m,k)|^2. \quad (3)$$

The values of variance (3) for common distributions belonging to the Cohen class are presented in Fig. 1. In this figure, as well as in all further numerical illustrations, we used the following notations:

- Born–Jordan distribution $c(\Theta,\tau) = \sin(\Theta\tau/2)/(\Theta\tau/2)$
- Choi–Williams distribution $c(\Theta,\tau) = e^{-\Theta^2\tau^2/\sigma^2}$
- sinc distribution $c(\Theta,\tau) = \mathrm{rect}(\Theta\tau/\alpha)$
- Butterworth distribution $c(\Theta,\tau) = 1/(1+(\Theta\tau/(\Theta_1\tau_1))^4)$
- optimal auto-term distribution $c(\Theta,\tau) = e^{-|\Theta\tau|/\sigma}$ [5]
- pseudo Wigner distribution with the Hanning window.

Kernels are given in the analog ambiguity domain. Discretization is done taking the range $|\Theta| \le \sqrt{\pi N}$ and $|\tau| \le \sqrt{\pi N}$ with $N = 32$. Kernel $\varphi(m,k)$ is calculated as a Fourier transform $\varphi(m,k) = FT_\theta[c(\theta,k)]$, where $c(\theta,k)$ are samples of $c(\Theta,\tau)$ along $\tau$, and $\theta$ is discrete-domain frequency $\theta = \Theta(\pi/\sqrt{\pi N})$. In order to compare various distributions, their parameters $(\sigma,\alpha,\theta_1\tau_1)$ are chosen according to the results in [5]. In [1], it has also been shown that variance (3) is minimal (under the marginal conditions and time support constraint) for the pseudo Born–Jordan distribution, i.e., the pseudo Born–Jordan distribution is optimal with respect to

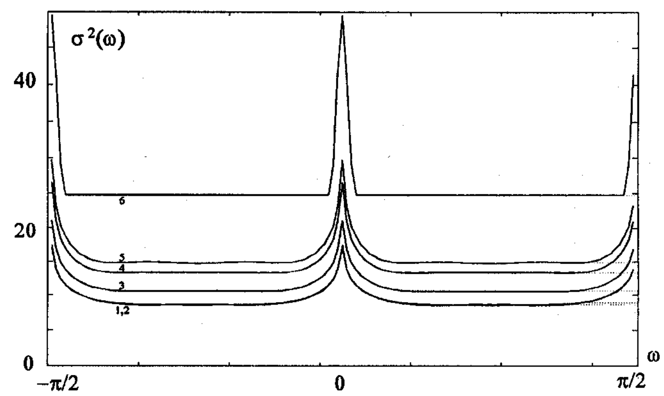


Fig. 2. Variance in the case of real noise for 1) Born–Jordan distribution, 2) optimal auto-term distribution, 3) Choi–Williams distribution, 4) Butterworth distribution, 5) sinc distribution, and 6) pseudo-Wigner distribution.

the variance under the described conditions. This is in complete agreement with the numerical data in Fig. 1. It is noteworthy that the distribution derived in [5] as optimal with respect to the auto-term form behaves in this case (as well as in the others that follow) almost exactly as does minimum variance distribution kernel (Born–Jordan kernel).

### B. Real Noise

We now consider a real noise $x(n)$ with variance $\sigma_x^2$. In this case, variance (2) contains all terms. After some straightforward manipulations (the same as in [1], [2], and [4]), it may be shown that the variance of the Cohen's class of distributions in the case of a real white Gaussian noise may be written as

$$\sigma^2(\omega)=\sigma_x^4 \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [|\varphi(m,k)|^2+\varphi(m,k)\varphi^*(m,-k)e^{-j4\omega k}]. \tag{4}$$

The above variance consists of two parts. One part (frequency independent) is the same as in (3). Thus, by minimizing (3), we also minimize this frequency-independent part of the variance. The other part of the variance is the sum over $m$ of the Fourier transform of $\varphi(m,k)\varphi^*(m,-k)$ over $k$. For distributions that are symmetric with respect to $k$, it holds that $\varphi(m,k) = \varphi(m,-k)$. This is the case for all known reduced-interference distributions. The Fourier transform is therefore applied to the positive and even function $|\varphi(m,k)|^2$. The transform's maximal value is reached at $\omega = 0$, and $\omega = n\pi/2 (n = \pm 1, \pm 2, \cdots)$. Accordingly

$$\max\{\sigma^2(\omega)\} = 2\sigma_x^4 \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2. \tag{5}$$

Thus, by minimizing (3), we minimize the maximal value of the second (frequency-dependent) part of variance (5) as well. This means that the conclusions drawn in [1] remain valid in the case of the real noise. The estimator variances, for the commonly used distributions, are presented in Fig. 2 for real noise.

### C. Analytic Noise

Commonly, in the numerical implementation of the quadratic distributions, an analytic part of signal is used rather than the signal itself. Here, we present the noise analysis if the noise is analytic. It may be written as $x_a(n) = x(n)+jx_h(n)$, where $x_h(n)$ is the Hilbert transform of $x(n)$. The autocorrelation function of analytic noise $x_a(n)$ is given by $R_{x_a x_a}(k) = 2(R_{xx}(k)+jR_{xx}(k)*h(k))$, where $h(k)$ is the impulse response of the Hilbert transform. The spectral power density of $x_a(n)$ for the white noise $x(n)$ is $S_{x_a x_a}(\omega) = 2\sigma_x^2 U(\omega), |\omega| < \pi$, where $U(\omega)$ is the unite step function. From the results in [1], [4], and (2) and using the fact that $R_{x_a^* x_a}(k) = R_{x_a x_a^*}(k) = 0$, the estimator variance is

$$\sigma^2(\omega) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \cdot \varphi(m_1,k_1)\varphi^*(m_2,k_2) \times [R_{x_a x_a}(m_1-m_2+k_1-k_2) \times R^*_{x_a x_a}(m_1-m_2-k_1+k_2)]e^{-j2\omega(k_1-k_2)}. \tag{6}$$

After some transformations (see the Appendix), we get the variance in the following form:

$$\sigma^2(\omega) = \frac{\sigma_x^4}{2\pi^2} \int_{-\pi}^{\pi} \int_{-|2\omega-\xi|}^{|2\omega-\xi|} |\Psi(\theta,\xi)|^2 \, d\xi \, d\theta \quad \text{for } |\omega| \le \frac{\pi}{2} \tag{7}$$

where $\Psi(\theta,\omega)$ is the kernel function in the frequency–frequency domain $\Psi(\theta,\omega) = FT_{m,k}[\varphi(m,k)]$, and the integration limits are of module $2\pi$ form. The kernel $\Psi(\theta,\xi)$ is mainly concentrated at and around the $(\theta,\xi)$ origin and $\xi = 0$ axis; see Fig. 3. With this in mind, as well as the fact that $|\Psi(\theta,\xi)|^2$ is always a positive function, we may easily conclude that the maximal value will be obtained for $|\omega| = \pi/2$. This value is very close to the value obtained by the integration over entire region $\theta,\xi \in (-\pi,\pi]$. The difference is equal to the integral outside the rhombus shown in Fig. 3. For example, for the Born–Jordan distribution, all values of $|\Psi(\theta,\xi)|^2$ outside the integration region for $|\omega| = \pi/2$ are less than $0.002\max\{|\Psi(\theta,\xi)|^2\}$, and thus, the difference of the integral over this region and the entire $(\theta,\xi)$ plane is of 1% order. This is shown in Fig. 4, where the variance (7) is depicted for various distributions along with the values obtained by the integration over the entire $(\theta,\xi)$ region (see the dotted lines on the right-hand side). According to the previous analysis, we may conclude that by minimizing (3), we also minimize the maximal value of (7), which is approximately equal to

$$\max\{\sigma^2(\omega)\} \cong \frac{\sigma_x^4}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Psi(\theta,\xi)|^2 \, d\xi \, d\theta = 2\sigma_x^4 \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2, \quad \text{at } |\omega| = \frac{\pi}{2}. \tag{8}$$

The above equation is obtained using Parseval's theorem, as well as the fact that $\Psi(\theta,\omega)$ is a two-dimensional (2-D) FT of the kernel function in $(m,k)$ domain. From (8) and the above analysis, we may conclude that the kernel that minimizes (3) also minimizes the maximal value of (7), which is given by (8). This further means that the Born–Jordan kernel remains optimal, under the assumed conditions, with respect to the variance maximal value.

Since the Born–Jordan distribution appears to be a key distribution from the point of noise influence, its variance values (3), (4), and (7) are checked statistically and presented in Fig. 5. Averaging is done over a set of the distribution values calculated at 2000 time instants with the same numerical data as in Section II-A. Agreement with theoretical data is very high.

## III. On the Noisy Signals

The analysis of the noise influence in the case of deterministic signals corrupted by noise is very difficult and highly signal dependent. This is the reason why reasonable results may be obtained only under some constraints imposed on the considered signals. In [2], Amin has shown that the results from [1] may be easily applied to the case of noisy frequency-modulated (FM) signals. In this section, we will generalize the results and conclusions from Section II, considering real noisy signals as well as analytic noisy signals. Let us assume a deterministic signal $f(n)$ in additive white Gaussian noise $\nu(n)$, i.e., $x(n) = f(n) + \nu(n)$. The variance of the noise will be denoted by $\sigma_\nu^2$. In this case, it can be easily shown [2], [4] that the distribution

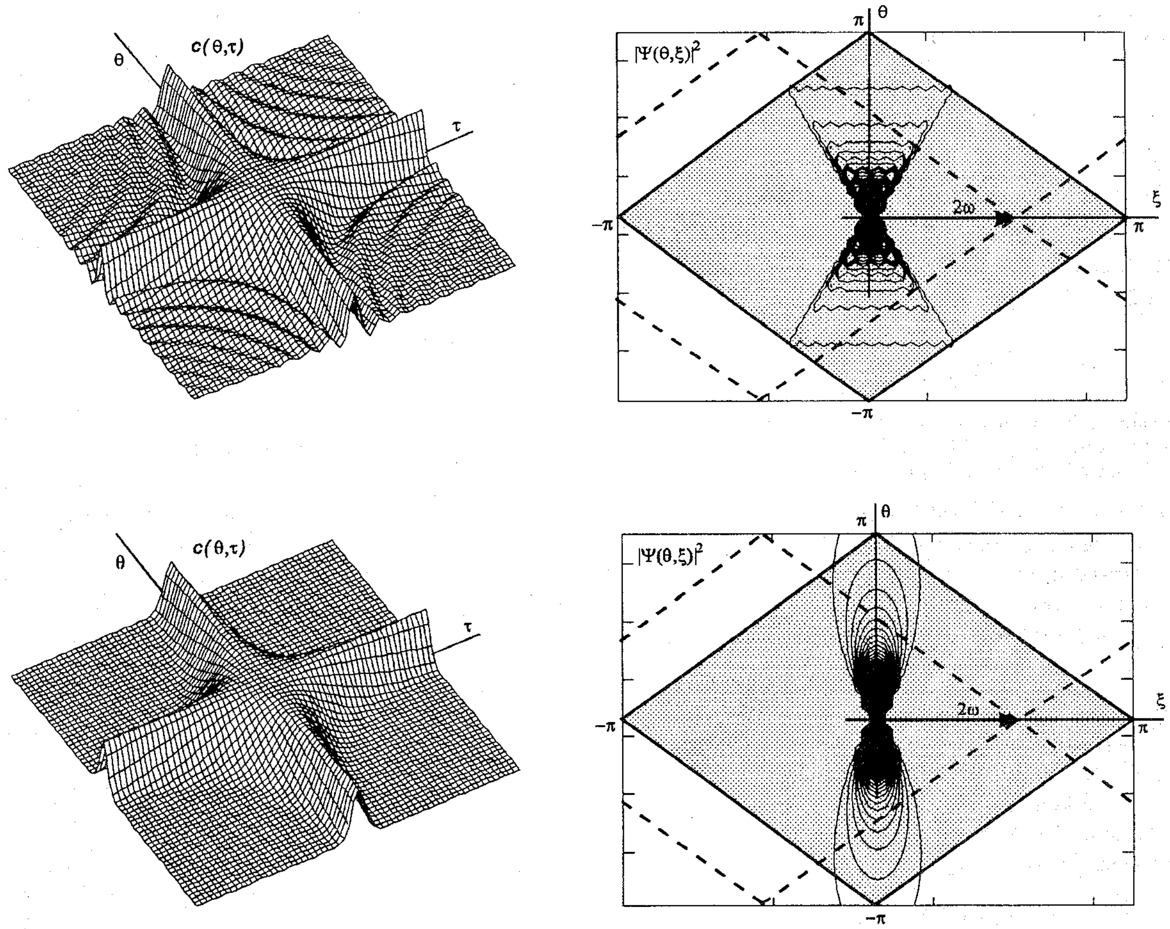


Fig. 3. Illustration of the variance calculation in the case of analytic noise for the Born–Jordan distribution (top) and Choi–Williams distribution (bottom). The contours of $|\Psi(\theta,\xi)|^2$ (right hand-side) are plotted at each $0.002\max\{|\Psi(\theta,\xi)|^2\}$.

estimator's variance consists of two components:

$$\sigma^2(\omega)=\sigma^2_{f\nu}(\omega)+\sigma^2_{\nu\nu}(\omega). \tag{9}$$

The component $\sigma^2_{f\nu}(\omega)$ depends on both the signal and additive noise, whereas the other component $(\sigma^2_{\nu\nu}(\omega))$ depends on the noise only. The later is exactly equal to the variance described in Section II for all three cases and is given by (3), (4) and (7) for complex, real, and analytic noise, respectively (replacing $\sigma_x^2$ by $\sigma_\nu^2$). Therefore, we will focus our attention only on the signal dependent component $\sigma^2_{f\nu}(\omega)$, which can be written in the form

$$\begin{aligned}\sigma^2_{f\nu}(\omega)=&\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\\ &\cdot\varphi(m_1,k_1)\varphi^*(m_2,k_2)\\ &\times[f(n+m_1+k_1)f^*(n+m_2+k_2)\\ &\cdot R^*_{\nu\nu}(m_1-m_2-k_1+k_2)\\ &+f^*(n+m_1-k_1)f(n+m_2-k_2)\\ &\cdot R_{\nu\nu}(m_1-m_2+k_1-k_2)\\ &+f(n+m_1+k_1)f(n+m_2-k_2)\\ &\cdot R_{\nu^*\nu}(m_1-m_2-k_1-k_2)\\ &+f^*(n+m_1-k_1)f^*(n+m_2+k_2)\\ &\times R_{\nu\nu^*}(m_1-m_2+k_1+k_2)]e^{-j2\omega(k_1-k_2)}.\end{aligned} \tag{10}$$

In this section (as in [2]), we will minimize the mean value of variance $\sigma^2(\omega)$

$$\overline{\sigma^2(\omega)}=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sigma^2(\omega)\,d\omega. \tag{11}$$

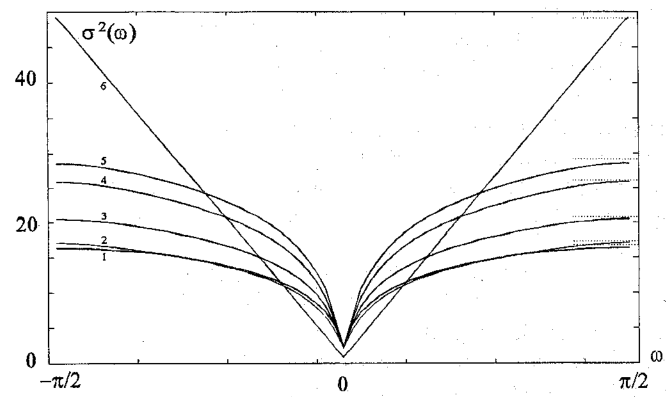


Fig. 4. Variance in the case of analytic noise for 1) Born–Jordan distribution, 2) optimal auto-term distribution, 3) Choi–Williams distribution, 4) Butterworth distribution, 5) sinc distribution, and 6) pseudo-Wigner distribution. Dotted lines at the right-hand side represent the kernel energy values.

The mean value will be used, rather than the exact variance value, since exact variance analysis would require a complete knowledge about the signal $f(n)$ and would be completely signal dependent and, consequently, inappropriate.

### A. Complex Noise

Assume, as in Section II-A, that the complex noise $\nu(n)$ contains independent real and imaginary parts, with variances $\sigma_\nu^2/2$. The mean variance $\overline{\sigma^2_{f\nu}(\omega)}$ is derived in [2] for the case of FM signals

$f(n) = Ae^{j\phi(n)}$ as

$$\overline{\sigma^2_{f\nu}(\omega)} = 2A^2\sigma^2_\nu \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2 \tag{12}$$

and consequently, the mean variance of the Cohen class of distributions [2] becomes

$$\overline{\sigma^2(\omega)} = (2A^2 + \sigma^2_\nu)\sigma^2_\nu \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2. \tag{13}$$

According to this expression, it is easy to conclude that by minimizing (3), we minimize (13). Thus, as shown in [2], the conclusions from [1] remain valid for the mean variance value in the case of complex noisy FM signals as well.

### *B. Real Noise*

In this case, the mean value of the variance in (10) takes the form

$$\begin{aligned}\overline{\sigma^2_{f\nu}(\omega)} = {} & \overline{\sigma^2_{f\nu}(\omega)}_{|\text{complex signal+noise}} \\ & + \sigma^2_\nu \sum_{k=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} \\ & \cdot \varphi(m_1,k)\varphi^*(m_2,k) \\ & \times [f(n+m_1+k)f(n+m_2-k) \\ & \cdot \delta(m_1-m_2-2k) \\ & + f(n+m_1-k)f(n+m_2+k) \\ & \delta(m_1-m_2+2k)].\end{aligned} \tag{14}$$

Assuming that the signal absolute value is always less or equal to $A, |f(n)| \leq A$

$$\begin{aligned}|\overline{\sigma^2_{f\nu}(\omega)}| \leq {} & |\overline{\sigma^2_{f\nu}(\omega)}|_{\text{complex signal+noise}} \\ & + A^2\sigma^2_\nu \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \\ & \cdot [|\varphi(m+2k,k)\varphi^*(m,k)| \\ & + |\varphi(m-2k,k)\varphi^*(m,k)|]\end{aligned} \tag{15}$$

or, finally[1] with $\varphi(m,k) = \varphi(m,-k)$

$$|\overline{\sigma^2_{f\nu}(\omega)}| \leq 4A^2\sigma^2_\nu \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2. \tag{16}$$

We have also searched for a more precise expression than (16) since the minimization has more sense if applied to closer expression to the exact one (the inequality in footnote 1 may be far from the equality for common distribution kernels). Note that in the second part of variance (15), we have $\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} [|\varphi(m+2k,k)\varphi^*(m,k)| + |\varphi(m-2k,k)\varphi^*(m,k)|]$, which, for $\varphi(m,k) = \varphi(m,-k)$, simplifies to $2\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m-2k,k)\varphi^*(m,k)|$. Knowing that $\varphi(m,k)$ is mainly concentrated at the origin and around the $k$ ($m=0$) axis (for all reduced interference distributions and Wigner distribution), we may conclude that

$$\begin{aligned} & 2\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m-2k,k)\varphi^*(m,k)| \\ & \quad \cong 2\sum_{m=-\infty}^{\infty} |\varphi(m,0)|^2 = 2|\varphi(0,0)|^2.\end{aligned} \tag{17}$$

For all distributions, satisfying frequency marginal property, this is a distribution-independent constant. Approximation (17) has been checked out for all considered distributions and found that the error is less than 1.5% of $2\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2$. Thus, we may write the upper bound of the variance for the real noisy signal as

$$\overline{\sigma^2_{f\nu}(\omega)} \leq 2A^2\sigma^2_\nu \left(\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2 + |\varphi 0,0)|^2\right). \tag{18}$$

The equality sign [with approximation (17)] holds for signal $f(n) = A$. The conclusion is the same as the earlier conclusion; minimization of the maximal possible value of (18) is achieved by minimizing $\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi(m,k)|^2$, which again reduces to the very well-studied factor and results presented in [1] and [2].

[1]For any function $\varphi(m,k)$, using the Schwartz inequality, we have

$$\sum_{m=-\infty}^{\infty} |\varphi(m+a,k)||\varphi(m,k)| \leq \sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2$$

where equality holds for $|\varphi(m+a,k)| = c|\varphi(m,k)|$, with $a$ and $c$ being arbitrary constants.

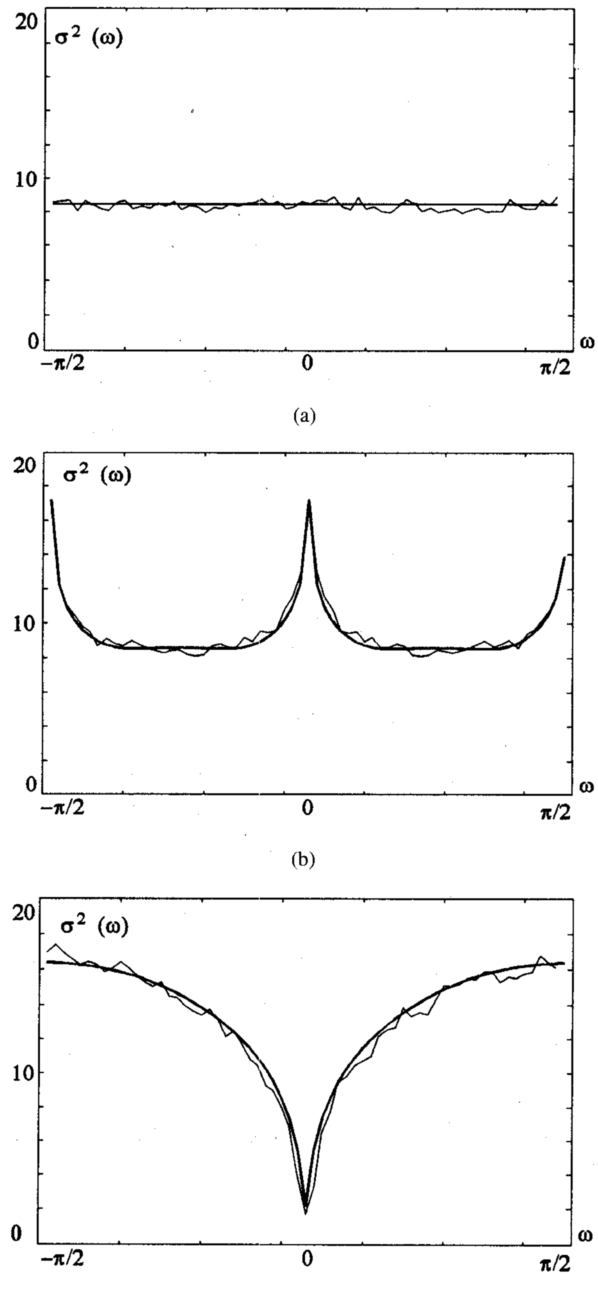


Fig. 5. Variances in the Born–Jordan distribution obtained theoretically (thick lines) and statistically (thin lines) for (a) complex noise with independent real and imaginary parts, (b) real noise, and (c) analytic noise.

### *C. Analytic Noise*

The analytic part of noise will be denoted by $\nu_a$. According to (10) and the fact that $R_{\nu_a\nu_a^*}(m) = R_{\nu_a^*\nu_a}(m) = 0$, the mean of the

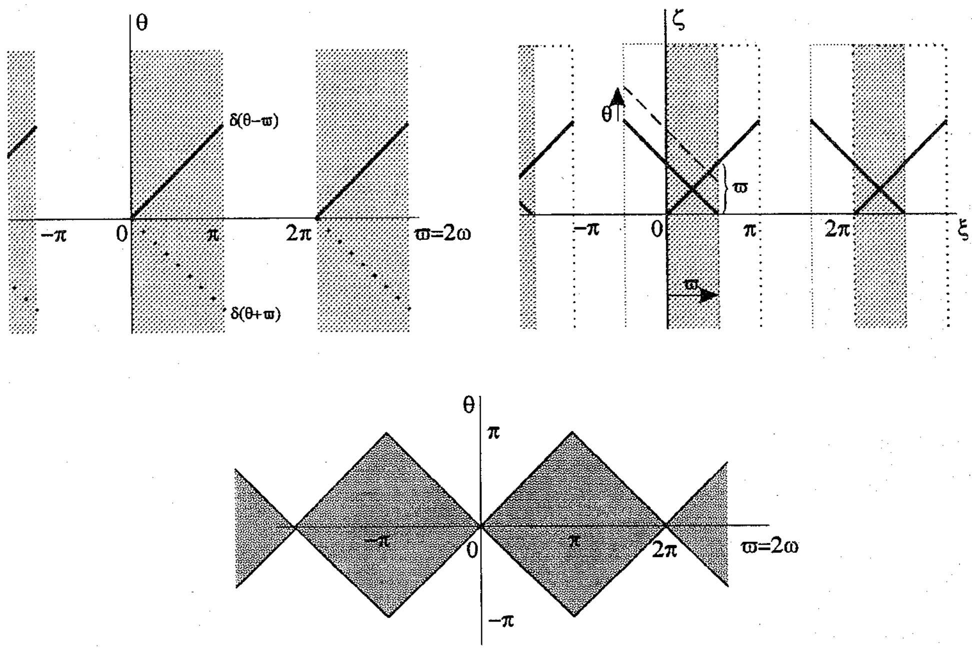


Fig. 6. Illustration of the integration domain in the case of analytic noise.

signal-dependent part of the variance is given by

$$\begin{aligned}\overline{\sigma^2_{f\nu}(\omega)} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} &\{\underline{\varphi(m,k)f(n+m+k)} \\ &\times \underline{\underline{[\varphi(m,k)f(n+m+k) *_m R_{\nu_a\nu_a}(m)]^*}} \\ &+ \underline{\varphi(m,k)f^*(n+m-k)} \\ &\times \underline{\underline{[\varphi(m,k)f^*(n+m-k) *_m R^*_{\nu_a\nu_a}(m)]^*}}\} \quad (19)\end{aligned}$$

where $*_m$ denotes a convolution over $m$. Using Parseval's theorem[2] on both terms (with underlined forms as entities) after some straightforward manipulations, and using the facts that

$$\begin{aligned}&|c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})|^2 \\ &\quad = |[c^*(-\theta,-k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})]^*|^2\end{aligned}$$

and $S^*_{\nu_a\nu_a}(\theta) = S_{\nu_a\nu_a}(\theta)$, we get the mean of variance $\sigma^2_{f\nu}(\omega)$ in the following form:

$$\overline{\sigma^2_{f\nu}(\omega)} = \sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi}|c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})|^2 S_{\nu_a\nu_a}(\theta)\frac{d\theta}{\pi} \quad (20)$$

where $F(e^{j\omega}) = FT[f(n)]$, and $c(\theta,k) = FT_m[\varphi(m,k)]$ is the kernel in the ambiguity $(\theta,k)$ domain.

Applying the Schwartz inequality[3] on the convolution over $\theta$ and using

$$\int_{-\pi}^{\pi}|c(\theta,k)|^2\,d\theta = 2\pi\sum_{m=-\infty}^{\infty}|\varphi(m,k)|^2, \quad \text{and}$$

$$\frac{1}{2\pi}\int_0^{\pi}\int_{-\pi}^{\pi}|F(e^{j(\theta-\mu)})|^2\,d\theta\,d\mu = \pi E_f$$

as well as

$$S_{\nu_a\nu_a}(\theta) = 2\sigma^2_\nu U(\theta) \quad \text{for } |\theta| < \pi$$

we get

$$\overline{\sigma^2_{f\nu}(\omega)} \le 2\sigma^2_\nu E_f\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi(m,k)|^2. \quad (21)$$

This is a general expression, which for some signals may not be a close approximation. More specific variance forms may be obtained for specific signals. Such signals will be considered in the sequel. Since $f(n)$ is analytic, $F(e^{j\theta}) = 0$ for $\theta < 0$. Assuming that $c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})$ is completely concentrated in $\theta \ge 0$, we get, for analytic signals of the form $f(n) = Ae^{j\phi(n)}$

$$\overline{\sigma^2_{f\nu}(\omega)} = 4A^2\sigma^2_\nu\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|\varphi(m,k)|^2 \quad (22)$$

using

$$\begin{aligned}&|c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})|^2 \\ &\quad = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\varphi(m_1,k)\varphi^*(m_2,k) \\ &\qquad \times f(n+m_1+k)f^*(n+m_2+k)e^{-j\theta(m_1-m_2)}\end{aligned}$$

and the fact that integration over $\theta$ from 0 to $\pi$, in this case, is equal to the integration from $-\pi$ to $\pi$. Note that the previous assumption is close to the real situation for the reduced interference distributions when the signal's Fourier transform $F(e^{j\theta})$ is not near the $\theta$ axis (in order to avoid convolution values for $\theta < 0$). We should mention that we cannot completely avoid convolution values for $\theta < 0$, at least for $k = 0$ when $c(\theta, 0) = 1$. Having this in mind, we would get the mean variance as (22) minus $A^2\sigma^2_\nu$. However, this factor is significantly smaller than (22), and therefore, we do not make any significant error leaving (22) as it stands. In order to determine the bounds, including the cases when a significant part of the convolution energy is in the region $\theta < 0$, consider a simple signal form $f(n) = Ae^{j\omega_0 n}$. In this case, $|c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})|^2 = A^2|c(\theta-\omega_0,k)|^2$. If $\omega_0 \to +0$, then the integrand in (20) tends to $A^2|c(\theta,k)|^2$. Having in mind that $|c(\theta,k)|^2$ is symmetric with respect to $\theta$, we get that $\overline{\sigma^2_{f\nu}(\omega)}$, in this case, is equal to the half of the value given by (22).

[2] The 1-D Parseval theorem follows:

$$\sum_{k=-\infty}^{\infty}x(k)y^*(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})\,d\omega$$

where $X(e^{j\omega}) = FT_k[x(k)]$ and $Y(e^{j\omega}) = FT_k[y(k)]$.

[3] The Schwartz inequality follows:

$$\left|\int_a^b z(t)w(t)\,dt\right|^2 \le \int_a^b|z(t)|^2\,dt\int_a^b|w(t)|^2\,dt.$$

The same result will be obtained for $\omega_0 \to \pi - 0$. The maximal value will be obtained for $\omega_0 \to \pi/2$ when it is approximately equal to (22). Thus, depending on the analytic signal form, we have

$$2A^2\sigma_\nu^2 \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2 \le \overline{\sigma_{f\nu}^2(\omega)} \le 4A^2\sigma_\nu^2 \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2. \tag{23}$$

We have seen that the upper limit (22) is true for any FM analytic signal, whereas the lower limit holds with the assumption that the part of energy of $c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})$ located in the region $\theta \in [-\pi, 0)$ is less or equal to the one in $\theta \in [0,\pi)$. This is a reasonable assumption since $F(e^{j\theta})$ exists only for $\theta \in [0,\pi)$, and $c(\theta,k)$ is symmetric around the $\theta$ axis.

This means that in minimizing (3), we also minimize the maximal possible value of the mean of the signal-dependent part of variance in the analytic noisy signal case (21) or the bounds within which the mean variance may take its values for different FM signal forms (23).

It is interesting that for the Wigner distribution, the energy of $c(\theta,k) *_\theta (F(e^{j\theta})e^{j\theta(n+k)})$ is always symmetric with respect to $\theta$ since $c(\theta,k) = 1$. Thus

$$\overline{\sigma_{f\nu}^2(\omega)} = 2A^2\sigma_\nu^2 \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} |\varphi(m,k)|^2$$

for any signal of the form $f(n) = Ae^{j\phi(n)}$. In addition, from (20), we may easily get the exact average variance for the Wigner distribution for any analytic signal as $\overline{\sigma_{f\nu}^2(\omega)} = 2\sigma_\nu^2 E_f$, where $E_f$ is the energy of an arbitrary analytic signal $f(n)$ (for $f(n) = Ae^{j\phi(n)}$, these two expressions are equivalent, keeping in mind that for the Wigner distribution, $\varphi(m,k) = \delta(m)$ within a bounded region in the $(m,k)$ domain [4]).

## IV. Conclusion

In this correspondence, we have shown that the results for minimization of the time-frequency distributions' variance, obtained for the complex noise and noisy signals with independent real and imaginary noise parts, may be directly applied in the other two very important cases of real and analytic noise and noisy signals since the key factor in minimization (the energy of a time-frequency distribution kernel function) remains the same in all of these cases.

## Appendix

Variance (6), using substitutions $k_1 - k_2 = k, k_1 = k_1$ and $m_1 - m_2 = m, m_1 = m_1$, may be written as

$$\sigma^2(\omega) = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} [\varphi(m,k) **_{m,k} \varphi^*(-m,-k)] \times R_{x_a x_a}(m+k) R^*_{x_a x_a}(m-k) e^{-j2\omega k} \tag{24}$$

where $**_{m,k}$ is a 2-D convolution over $m$ and $k$. Considering the above expression as a 2-D Fourier transform over $m$ and $k$, at $\theta = 0$ and $\varpi = 2\omega$ with

$$\Psi(\theta,\omega) = FT_{m,k}[\varphi(m,k)],$$
$$\Psi^*(\theta,\omega) = FT_{m,k}[\varphi^*(-m,-k)], \quad \text{and}$$
$$\Omega(\theta,\omega) = FT_{m,k}[R_{x_a x_a}(m+k) R^*_{x_a x_a}(m-k)]$$

we get

$$\sigma^2(\omega) = \{|\Psi(\theta,\varpi)|^2 **_{\varpi,\theta} \Omega(\theta,\varpi)\}_{|\theta=0,\varpi=2\omega}. \tag{25}$$

Since the spectral power density of analytic noise has the value

$$S_{x_a x_a}(\omega) = FT_k[R_{x_a x_a}(k)] = 2\sigma_x^2 U(\omega), \quad \text{for } |\omega| < \pi$$

we get

$$\Omega(\theta,\varpi) = 16\pi^2\sigma_x^4 [U(\varpi)\delta(\varpi-\theta) **_{\varpi,\theta} U(\varpi)\delta(\varpi+\theta)]. \tag{26}$$

Convolution (26) is illustrated in Fig. 6. Note that

$$\iint_D \delta(\varpi-\theta)\delta(\varpi+\theta)\, d\varpi\, d\theta = \tfrac{1}{2} \text{ if } (\theta,\varpi) = (0,0)$$

belongs to the integration domain $D$ and 0 otherwise. According to this result and Fig. 6, (25) assumes the form

$$\sigma^2(\omega) = \frac{\sigma_x^4}{2\pi^2} \int_{-\pi}^{\pi} \int_{-|2\omega-\xi|}^{|2\omega-\xi|} |\Psi(\theta,\xi)|^2\, d\xi\, d\theta \quad |\omega| \le \frac{\pi}{2} \tag{27}$$

where the integration limits should be considered as module $2\pi$ quantities; see Fig. 6.

### *Variance Mean Value*

The mean value of variance (6) is given by

$$\begin{aligned}\overline{\sigma^2(\omega)} &= \sum_{k=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} \varphi(m_1,k)\varphi^*(m_2,k) \times [R_{x_a x_a}(m_1-m_2) R^*_{x_a x_a}(m_1-m_2)] \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \varphi(m,k)[\varphi(m,k) *_m \cdot (R^*_{x_a x_a}(m) R_{x_a x_a}(m))]^*\end{aligned} \tag{28}$$

and using Parseval's theorem, we get

$$\overline{\sigma^2(\omega)} = \frac{\sigma_x^4}{\pi^2} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} |c(\theta,k)|^2 (\pi - |\theta|)\, d\theta. \tag{29}$$

Since the factor $\int_{-\pi}^{\pi} |c(\theta,k)|^2|\theta|\, d\theta$ may have significant value with respect to $\int_{-\pi}^{\pi} |c(\theta,k)|^2\, d\theta$, this expression could not be reduced to the previously studied forms. However, this very simple expression may be used to check the results. For example, for the pseudo-Wigner distribution, this factor is equal to $\sigma_x^4 E_w$, where $E_w$ is the energy of the window in the $k$ domain. For the unite variance noise and the Hanning window of width $N = 64$, we get $\overline{\sigma^2(\omega)} = 24$; see Fig. 4.


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## References


[1] S. B. Hearon and M. G. Amin, "Minimum-variance time-frequency distributions kernels," *IEEE Trans. Signal Processing*, vol. 43, pp. 1258–1262, May 1995.

[2] M. G. Amin, "Minimum-variance time-frequency distribution kernels for signals in additive noise," *IEEE Trans. Signal Processing*, vol. 44, pp. 2352–2356, Sept. 1996.

[3] D. Wu and J. M. Morris, "Discrete Cohen's class of distributions," in *Proc. IEEE Symp. TFTSA*, Philadelphia, PA, Oct. 1994, pp. 532–535.

[4] LJ. Stanković and S. Stanković, "On the Wigner distribution of discrete-time noisy signals with application to the study of quantization effects," *IEEE Trans. Signal Processing*, vol. 42, pp. 1863–1867, July 1994.

[5] LJ. Stanković, "Auto-term representation by the reduced interference distributions: A procedure for kernel design," *IEEE Trans. Signal Processing*, vol. 44, pp. 1557–1564, June 1996.