S-class of time–frequency distributions

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Abstract: A new general class of distributions (S-class of distributions) for time–frequency signal analysis is proposed. This class is derived by generalising recently defined S-distribution. It is possible to define the S-counterpart distribution for each known distribution from the Cohen class, such that some of the performances may be improved. This class of distributions may be treated as a variant of the author's L-class of distributions, but it may satisfy unbiased energy conditions, time marginal as well as the frequency marginal in the case of asymptotic signals. A method for the realisation of the S-distribution which will be, in the case of multicomponent signals, equal to the sum of S-distributions of each component separately, is presented. Theory is illustrated by examples.

1 Introduction

Time–frequency analysis has been intensively studied lately. The whole variety of tools for time–frequency analysis, mainly rendered in the form of energy distributions in the time–frequency plane, has been proposed (an extensive list may be found in review papers [1, 2]). The oldest and most widely used method for time–frequency signal analysis is based on a straightforward extension of the Fourier transform, by using a window function to extract the signal's spectral content at and around a given time instant. It is referred to as the short time Fourier transform and belongs to linear signal transformations. Many performances of the time–frequency representation may be improved using quadratic distributions. The first quadratic representation was based on the Wigner distribution (originally defined in quantum mechanics and introduced into signal processing by Ville). Since then, many other quadratic distributions have been defined. Cohen has shown that all shift-covariant quadratic time–frequency distributions are just special cases of a general class of distributions, obtained for a particular choice of an arbitrary function (kernel), [1–4]. Out of the Cohen class, the Wigner distribution is the only one (with a signal independent kernel) which produces the ideal energy concentration along instantaneous frequency for the linear frequency modulated signals, [4–7].

To improve distribution concentration, when the instantaneous frequency is a polynomial function of time, polynomial Wigner distributions are proposed [8]. A similar idea for improving the distribution concentration for signals whose phase is polynomial up to the fourth order, was presented in [9]. To improve distribution concentration for a signal with an arbitrary nonlinear instantaneous frequency, The L-Wigner distribution was proposed and studied in [6, 7, 9, 10]. This distribution is generalised to the L-class of distributions in [7, 11]. The polynomial Wigner distribution, as well as the L-Wigner distributions, are closely related to the time-varying higher-order spectra [10, 12, 13]. They do not preserve the usual marginal properties [1, 2, 6], but they do satisfy the generalised forms of the marginals. For example, the time marginal in the L-Wigner distribution is the generalised power $|x(t)|^{2L}$, rather than $|x(t)|^2$. Here, we will present a new S-class of distributions which may achieve high concentration at the instantaneous frequency, as high as distributions from the L-class, while at the same time satisfying the energy unbiased condition, time marginal and, for asymptotic signals, frequency marginal property.

2 S-distribution

The scaled variant of the L-Wigner distribution (S-distribution) of a signal $x(t)$ is defined by [14, 15].

$$SD_L(t,\omega) \doteq \int_\tau x^{[L]}\left(t+\frac{\tau}{2L}\right) x^{[L]*}\left(t-\frac{\tau}{2L}\right) e^{-j\omega\tau} d\tau \qquad (1)$$

where $x^{[L]}(t)$ is the modification of $x(t)$ obtained by multiplying the phase function by L, while keeping the amplitude unchanged

$$x^{[L]}(t) \doteq A(t)e^{jL\phi(t)} \qquad (2)$$

The Wigner distribution is obtained from eqn. 1 with $L = 1$. All integrals are from $-\infty$ to ∞.

The S-distribution satisfies the time marginal property, the unbiased energy condition and, for asymptotic signals, the frequency marginal property. The integral of $SD_L(t, \omega)$ over ω is equal to signal power $|x(t)|^2$

$$\frac{1}{2\pi}\int_\omega SD_L(t,\omega)d\omega$$
$$= \frac{1}{2\pi}\int_\omega\int_\tau x^{[L]}\left(t+\frac{\tau}{2L}\right) x^{[L]*}\left(t-\frac{\tau}{2L}\right) e^{-j\omega\tau} d\tau d\omega$$
$$= |x(t)|^2 \qquad (3)$$

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From eqn. 3, it is obvious that the unbiased energy condition is satisfied, as well

$$\frac{1}{2\pi}\int_t\int_\omega SD_L(t,\omega)dtd\omega = \int_t |x(t)|^2 dt = E_x$$

where E_x is the energy of signal $x(t)$.

The frequency domain form of $SD_L(t, \omega)$ is

$$SD_L(t,\omega) = \frac{L}{2\pi}\int_\theta X_L\left(L\omega + \frac{\theta}{2}\right) X_L^*\left(L\omega - \frac{\theta}{2}\right) e^{j\theta t} d\theta \tag{4}$$

where $X_L(\omega)$ is the Fourier transform of $x^{[L]}(t)$. The integral of $SD_L(t, \omega)$ over time is

$$\int_t SD_L(t,\omega)dt = L|X_L(L\omega)|^2$$

According to the stationary phase method [16], we have

$$X_L(L\omega) \doteq \int_t x^{[L]}(t)e^{-jL\omega t}dt$$

$$= \int_t A(t)e^{jL\phi(t)}e^{-jL\omega t}dt$$

$$= A(t_0)e^{jL[\phi(t_0)-\omega t_0]}\sqrt{\frac{2\pi j}{L\phi''(t_0)}} \tag{5}$$

The above relation holds for any signal with continuous $A(t)$ if $L \to \infty$. For asymptotic signals [16, 17](signals whose amplitude variations are much slower than the phase variations $|A'(t)| << |\phi'(t)|$) holds for any L. Note that t_0 is a function of ω defined by $\phi'(t_0) - \omega = 0$, with $\phi''(t_0) \neq 0$. From eqn. 5 it is easy to conduce that, for asymptotic signals

$$L|X_L(L\omega)|^2 = |X(\omega)|^2$$

meaning that the S-distributions, in this case, satisfy the frequency marginal property as well.

For asymptotic signals, according to eqn. 5, we may also write $\sqrt{[L/j]}X_L(L\omega) = X^{[L]}(\omega)$ meaning that the frequency domain form eqn. 4, for these signals, assumes a form dual to eqn. 1

$$SD_L(t,\omega) = \frac{1}{2\pi}\int_\theta X^{[L]}\left(\omega + \frac{\theta}{2L}\right) X^{*[L]}\left(\omega - \frac{\theta}{2L}\right) e^{j\theta t} d\theta \tag{6}$$

This relation shows that all properties valid in the time domain remain valid in the frequency domain, for asymptotic signals.

3 S-class of distributions

3.1 Definition

To generalise the S-distribution into a class of distributions, preserving the marginal properties, let us define an arbitrary two-dimensional function $S_L(t, \omega)$ with a two-dimensional Fourier transform $AG_L(\theta, \tau)$

$$S_L(t,\omega) \doteq \frac{1}{2\pi}\int_\theta\int_\tau AG_L(\theta,\tau)e^{-j\omega\tau + j\theta t}d\theta d\tau$$

$$AG_L(\theta,\tau) \doteq \frac{1}{2\pi}\int_t\int_\omega S_L(t,\omega)e^{j\omega\tau - j\theta t}dtd\omega \tag{7}$$

The integral of $S_L(t, \omega)$ over time and frequency is equal to signal $x(t)$ energy if and only if $AG_L(0, 0) = E_x$ (energy condition). The proof is evident from the second part of eqn. 7 and the uniqueness of the Fourier transform. Thus, if we know only one distribution satisfying $AG_L(0, 0) = E_x$, then any other function having a two-dimensional Fourier transform $MG_L(\theta, \tau) = c_L(\theta, \tau)AG_L(\theta, \tau)$ satisfies the unbiased energy condition if arbitrary function $c_L(\theta, \tau)$, called the kernel, has the unity value at the origin $c_L(0, 0) = 1$. Function $S_L(t, \omega)$ satisfies time and frequency marginal properties if

$$AG_L(\theta,0) = \frac{1}{2\pi}\int_t\left[\int_\omega S_L(t,\omega)d\omega\right]e^{-j\theta t}dt$$

$$= \int_t |x(t)|^2 e^{-j\theta t}dt$$

$$AG_L(0,\tau) = \frac{1}{2\pi}\int_\omega\left[\int_t S_L(t,\omega)dt\right]e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_t |X(\omega)|^2 e^{j\omega\tau}d\omega$$

It is evident that the marginal properties are determined by the values of $AG_L(\theta, \tau)$ along θ and τ axes only. This means that if function $S_L(t, \omega)$ satisfies marginal properties, then any other function with two-dimensional Fourier transform $MG_L(\theta, \tau) = c_L(\theta, \tau) AG_L(\theta, \tau)$, such that $c_L(\theta, 0) = 1$ and $c_L(0, \tau) = 1$, satisfies marginals as well.

A general function, satisfying the same marginal properties as $S_L(t, \omega)$ will be denoted by $SC_L(t, \omega)$ and referred to as the S-class of distribution. According to the above consideration, it is defined by

$$SC_L(t,\omega) = \frac{1}{2\pi}\int_\theta\int_\tau c_L(\theta,\tau)AG_L(\theta,\tau)e^{-j\omega\tau+j\theta t}d\theta d\tau$$

$$= \frac{1}{4\pi^2}\int_\theta\int_\tau\int_v\int_u c_L(\theta,\tau)S_L(u,v)e^{jv\tau - j\theta u}e^{-j\omega\tau + j\theta t}dudvd\theta d\tau \tag{8}$$

Taking the S-distribution eqn. 1 as the basic function for generalisation ($S_L(t, \omega) \equiv SD_L(t, \omega)$) in eqn. 8, we get the S-class in the form

$$SC_L(t,\omega) = \frac{1}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau)x^{[L]}\left(u + \frac{\tau}{2L}\right)x^{*[L]}\left(u - \frac{\tau}{2L}\right)e^{-j\omega\tau}e^{-j\theta(u-t)}dud\theta d\tau \tag{9}$$

For $L = 1$ this class of distributions reduces to the Cohen class [1, 2].

The distribution in eqn. 8 is an inverse two-dimensional Fourier transform of the product of $c(\theta, \tau)$ and $AG_L(\theta, \tau)$, thus it may be expressed as a two-dimensional convolution of the Fourier transform of $c_L(\theta, \tau)$, denoted by $\Pi_L(t, \omega)$ and $SD_L(t, \omega)$

$$SC_L(t,\omega) = \frac{1}{2\pi}\int_u\int_v \Pi_L(t-u,\omega-v)SD_L(u,v)dudv \tag{10}$$

All distributions from the S-class may be treated as smoothed versions of the S-distribution. Expressions in eqns. 9 and 10, as well as the complete theory presented in the paper, may be easily extended to the time-scale energy distributions [6, 18]

$$SC_L(t,a) = \frac{1}{2\pi}\int_u\int_v \Pi_L\left(\frac{t-u}{a}, \omega_0 - av\right) SD_L(u,v)dudv$$

where a is a scale factor $a \equiv \omega_0/\omega$.

For the asymptotic signals, using eqn. 6 of the S-distribution, we get the S-class in terms of the signal's Fourier transform

$$SC_L(t,\omega) = \frac{1}{4\pi^2}\int_\theta\int_v\int_\tau c_L(\theta,\tau)X^{[L]}\left(v+\frac{\theta}{2L}\right)X^{*[L]}\left(v-\frac{\theta}{2L}\right)e^{j\theta t}e^{-j\tau(\omega-v)}dvd\theta d\tau \quad (11)$$

3.2 Properties

Properties of distributions belonging to the S-class, will be listed in order as they appeared in [2]. Many of them may be proved in a straightforward manner, as in the Cohen class or the L-class of distributions [1, 2, 11], which is defined by

$$LD_L(t,\omega) = \frac{1}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau)x^L\left(u+\frac{\tau}{2L}\right)x^{*L}\left(u-\frac{\tau}{2L}\right)e^{-j\omega\tau}e^{-j\theta(u-t)}dud\theta d\tau$$

These properties will be given without proofs, or any additional explanation. Attention will be paid only to those for which the S-class behaves in a qualitatively different manner than the Cohen class.

P_1: A distribution from the S-class of distributions is real if its S-generalised autocorrelation function

$$SRA_L(t,\tau) = \frac{1}{2\pi}\int_\theta\int_u c_L(\theta,\tau)x^{*[L]}\left(u-\frac{\tau}{2L}\right)x^{[L]}\left(u+\frac{\tau}{2L}\right)e^{-j\theta(u-t)}d\theta du \quad (12)$$

is Hermitian, $SRA_L(t, \tau) = SRA_L^*(t, -\tau)$. This condition is satisfied for $c_L(\theta, \tau) = c_L^*(-\theta, -\tau)$.

P_2, P_3: The S-class of distributions is time and frequency shift invariant if the kernel $c_L(\theta, \tau)$ is not time (t) and frequency (ω) dependent.

P_4: Time marginal is satisfied for distributions with $c_L(\theta, 0) = 1$.

P_5: If $c_L(0, \tau) = 1$, then the frequency marginal is satisfied for asymptotic signals.

P_6: The time moments property is satisfied for distributions with $c_L(\theta, 0) = 1$.

P_7: The frequency moments property holds for asymptotic signals if $c_L(0, \tau) = 1$.

P_8: If distribution $SC_L(t, \omega)$ corresponds to $x(t)$, then $SC_L(at, \omega/a)$ is a distribution of $\sqrt{|a|}x(at)$, $a \neq 0$, provided that $c_L(a\theta, \tau/a) = c_L(\theta, \tau)$.

P_9: For signal $x(t) = A(t)\exp(j\phi(t))$, the mean frequency

$$\langle\omega\rangle_t = \frac{\int_\omega \omega SC_L(t,\omega)d\omega}{\int_\omega SC_L(t,\omega)d\omega}$$

is invariant with respect to L and it is equal to the instantaneous frequency $\phi'(t)$, if

$$c_L(\theta,0) = 1 \quad \text{and} \quad \left.\frac{\partial c_L(\theta,\tau)}{\partial\tau}\right|_{\tau=0} = 0$$

Here, we will show that not only is the mean frequency equal to the instantaneous frequency, but also that for any signal, we may get a distribution whose signal power is completely concentrated at the instantaneous frequency.

Frequency modulated signals representation: The ideal distribution, concentrated along instantaneous frequency is defined by $2\pi A^2\delta(\omega - \phi'(t))$ or by $A^2W(\omega - \phi'(t))$ if a finite time interval, determined by the window $w(\tau) = FT^{-1}\{W(\omega)\}$, is used. For signal $x(t) = Ae^{j\phi(t)}$, this form may be obtained in the Cohen class of distributions, only if the instantaneous frequency is a linear function $\phi'(t) = at + b$. Distribution which produces this concentration is the Wigner distribution (or pseudo Wigner distribution). If the instantaneous frequency variations are of a higher order than linear, then no distribution (with signal independent kernel) from the Cohen class can produce the ideal concentration.

Theorem 1: The S-class of distributions for $L \to \infty$ is equal to the ideal form $A^2(t)W(\omega - \phi'(t))$ for any signal $x(t) = A(t)e^{j\phi(t)}$ if derivatives of the phase function $\phi(t)$ are finite, $A(t)$ is continuous and $\lim_{L\to\infty} c_L(\theta, \tau) = w(\tau)$, where $w(\tau) = FT^{-1}\{W(\omega)\}$ is a finite duration window.

Proof: For a signal of the form $x(t) = A(t)e^{j\phi(t)}$, expanding $\phi(u \pm \tau/2L)$ into a Taylor series around u up to the third-order term, we get

$$SC_L(t,\omega) = \frac{A^2(t)}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau)e^{j\phi'(u)\tau} \times e^{j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\frac{\tau^3}{8}}e^{j\theta t-j\omega\tau-j\theta u}dud\theta d\tau \quad (13)$$

where τ_1, τ_2 are variables $0 \le |\tau_{1,2}| \le |\tau/2L|$. If $\phi^{(3)}(\tau)$ and $\phi^{(n)}(\tau)$, $n > 3$ are finite and variable τ may assume only finite values, then for a large L we have

$$\lim_{L\to\infty}\exp\left(j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\frac{\tau^3}{8}\right) = 1$$

and $A(t + \tau/2L)A(t - \tau/2L) \cong A^2(t)$, so we get

$$SC_L(t,\omega) \cong \frac{A^2(t)}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau) \times e^{j\phi'(u)\tau}e^{j\theta t-j\omega\tau-j\theta u}dud\theta d\tau \quad (14)$$

In this way the S-class of distributions locally linearises the instantaneous frequency characteristics. Eqn. 14 may be written in the form

$$SC_L(t,\omega) \cong A^2(t)\int_u \Pi_L(t-u,\omega-\phi'(u))du \quad (15)$$

where $\Pi_L(t, \omega)$ is a two-dimensional Fourier transform of $c_L(\theta, \tau)$. If $\lim_{L\to\infty} c_L(\theta, \tau) = w(\tau)$, then for large L $\Pi_L(t, \omega) = \delta(t)W(\omega)$ and $SC_L(t, \omega) = A^2(t)W(\omega - \phi'(t))$. This form corresponds to the ideal distribution concentration.

P_{10}: For asymptotic signals, the mean delay

$$\langle t\rangle_\omega = \frac{\int_t tSC_L(t,\omega)dt}{\int_t SC_L(t,\omega)dt}$$

is equal to the group delay $-d(\arg[X(\omega)])/d\omega$, if $c_L(0, \tau) = 1$ and $\partial c_L(\theta, \tau)/\partial\theta_{|\theta=0} = 0$. Proof is the same as in P_9 having in mind duality between eqns. 9 and 11. Note also that for asymptotic signals the group delay is an inverse function of instantaneous frequency $d(\arg[X(\omega)])/d\omega = \phi'^{-1}(\omega)$.

P_{11}: If a signal is time-limited to $|t| < T$, then $SC_L(t, \omega)$ is limited to the same time interval if $C_L(t, \tau) = FT_\theta\{c_L(\theta, \tau)\} = 0$ for $|t/\tau| > 1/(2L)$.

P_{12}: If an asymptotic signal is band-limited to $|\omega| < \omega_m$, then the S-class of distributions is band-limited to the same bandwidth if $C_L(\theta, \omega) = FT_\tau\{c_L(\theta, \tau)\} = 0$ for $|\omega/\theta| > 1/(2L)$.

P_{13}: The inner product of the S-class cross-distribution of signals $x(t)$ and $y(t)$ is equal to the product of energies of these signals multiplied by L: $\langle SC_{x,y}(t, \omega), SC_{x,y}(t, \omega)\rangle = L\langle x(t), x(t)\rangle \langle y(t), y(t)\rangle^*$, for $|c_L(\theta, \tau)|^2 = 1$.

P_{14}: If signals $x(t)$ and $h(t)$ are asymptotic then S-class distribution of the convolution of these signals $(x(t) * h(t))$ is equal to a convolution in time of distributions of each signal separately, for $c_L(\theta, \tau_1)c_L(\theta, \tau_1) = c_L(\theta, \tau_1 + \tau_2)$.

P_{15}: For the product of signals, $x(t)h(t)$, we have that, if $c_L(\theta_1, \tau)c_L(\theta_2, \tau) = c_L(\theta_1 + \theta_2, \tau)$, then the S-class of distributions is equal to a convolution in frequency of the distributions of each signal separately, $SC_{xh,L}(t, \omega) = SC_{x,L}(t, \omega) *_\omega SC_{h,L}(t, \omega)$.

P_{16}: If we replace an asymptotic signal with its scaled Fourier transform $\sqrt{[|c/2\pi|]}X(ct)$, $c \neq 0$, then its S-class of distributions is equal to $SC_L(-\omega/c, ct)$, for $c_L(-c\tau, \theta/c) = c_L(\theta, \tau)$.

P_{17}: The Fourier transform of an asymptotic signal $x(t)$ convolved with $\sqrt{[|c/2\pi|]}e^{jct^2/2}$ is equal to $\sqrt{j}X(\omega)e^{-j\omega^2/(2c)}$. S-class of distributions of this convolution, according to eqn. 11, is $SC_L(t - \omega/c, \omega)$, with $c_L(\theta, \tau + \theta/c) = c_L(\theta, \tau)$.

P_{18}: Multiplication of an arbitrary signal $x(t)$ by chirp signal $e^{jct^2/2}$ results in $SC_L(t, \omega - ct)$, if $c_L(\theta + c\tau, \tau) = c_L(\theta, \tau)$.

P_{19}: The order recursion relation holds for the S-class of distribution:

Theorem 2: For the unity amplitude signals, an Lth order distribution, belonging to the S-class, may be obtained from its $L/2$th order form if $c_L(\theta, \tau) = c_{L/2}(u, \tau/2)\, c_{L/2}(\theta - u, \tau/2)$ for any u.

Proof: The two-dimensional Fourier transform eqn. 7 of the S-distribution eqn. 1 is

$$AG_L(\theta,\tau) = \int_t x^{[L]}\left(t + \frac{\tau}{2L}\right) x^{[L]*}\left(t - \frac{\tau}{2L}\right) e^{-j\theta t} dt$$

For the unity amplitude signals $x^{[L]}(t) = x^L(t)$, thus we have

$$AG_L(\theta,\tau) = AG_{L/2}(\theta,\tau/2) *_\theta AG_{L/2}(\theta,\tau/2)$$

where $*_\theta$ is a convolution in θ. According to the theorem's kernel constraint, it follows that

$$MG_L(\theta,\tau) = MG_{L/2}(\theta,\tau/2) *_\theta MG_{L/2}(\theta,\tau/2)$$

Taking a two-dimensional Fourier transform of both sides, we get

$$SC_L(t,\omega) = \int_\lambda SC_{L/2}(t,\omega+\lambda) SC_{L/2}(t,\omega-\lambda)\frac{d\lambda}{\pi} \quad (16)$$

This result will be used in the realisation of the distributions belonging to the S-class, in the case of multicomponent signals, as well as to avoid signal oversampling.

Corollary: For the unity amplitude signals, any Lth order distribution may be expressed in terms of the $L/2$th order S-distribution.

Proof: Eqn. 16 is valid for the S-distribution. Inserting this relation into eqn. 10 we get any distribution expressed in terms of the $L/2$th order S-distribution.

Finally, let us emphasise that properties P_1, P_2, P_3, P_4, P_6, P_8, P_9, P_{11}, P_{13}, P_{15}, P_{18} and P_{19} may be satisfied for any signal, while the remaining ones may be fulfilled only for asymptotic signals.

4 Specific distributions

Some particular distributions belonging to the S-class will be briefly presented in this Section.

4.1 S-distribution

The S-distribution, which is the most important member of this class, is already presented in Section 2. Since it is taken as a basis for the generalisation, obviously its kernel is $c_L(\theta, \tau) = 1$, or for its pseudo form $c_L(\theta, \tau) = w_L(\tau)$. The properties and applications of the S-distribution are studied in [14, 15]. The numerical realisation will be described in the following Sections.

4.2 S-Rihaczek distribution

Distribution from the S-class corresponding to the Rihaczek distribution is defined as

$$SRD_L(t,\omega) \doteq \int_\tau x^{[L]}\left(t + \frac{\tau}{L}\right) x^{*[L]}(t) e^{-j\omega\tau} d\tau \quad (17)$$

The kernel for this distribution is $c_L(\theta, \tau) = e^{j\theta\tau/2L}$. For a frequency modulated signal $x(t) = A\exp(j\phi(t))$, with $\phi(t) = a + bt + ct^2/2$, we get

$$SRD_L(t,\omega) = A^2\delta(\omega - \phi'(t)) *_\omega FT\left\{e^{jc\tau^2/(2L)}\right\}$$
$$\lim_{L\to\infty} SRD_L(t,\omega) = 2\pi A^2\delta(\omega - \phi'(t))$$

The convergence toward the ideal concentration, in this case, is of the order of $1/L$, which is worse than in the S-distribution.

4.3 S-spectrogram and S-short time Fourier transform

The squared modulus of the S-short time Fourier transform (S-STFT) will be referred to as the S-spectrogram. It is defined by

$$SSP_L(t,\omega) \doteq \left| \int_\tau w_L(\tau) x^{[L]}\left(t + \frac{\tau}{L}\right) e^{-j\omega\tau} d\tau \right|^2 \quad (18)$$

We will focus attention only on the time and frequency resolutions of this distribution. First assume that signal $x(t)$ is short, concentrated at $t = 0$ into an interval $\Delta t \to 0$. If window $w_L(t)$ is time limited to $|t| < T/2$ (where $T >> \Delta t$), then the S-STFT is time limited to $|t| < T/(2L)$, i.e. its duration is $d = T/L$. If we now assume a sinusoidal signal $x(t) = \exp(j\omega_0 t)$ and the same window, we get $SSP_L(\omega, t) = |W_L(\omega - \omega_0)|^2$. For example, let the window be rectangular. The width of its Fourier transform $W_L(\omega)$ (the width of its main lobe) is $D = 4\pi/T$.

The product of durations d and D (the form of uncertainty principle in this case) is $dD = 4\pi/L$. This relation states that the S-STFT, with a given L, cannot be localised in the time–frequency plane with arbitrary small d and D simultaneously (representing the resolutions in time and frequency directions). But, the previous relation permits an important conclusion: By increasing L, the product dD can be made arbitrarily small meaning there are arbitrary high resolutions in both directions, simultaneously.

5 On the realisation

5.1 Direct method

The direct method is based on the straightforward application of a distribution definition. For the S-distribution, eqn. 1 with eqn. 2, signal $x(t)$ should be modified into $x^{[L]}(t)$, as well as oversampled L times with respect to the sampling interval in distribution with $L = 1$. The number of samples that are used for calculation may be kept unchanged. Regarding the last assumption, this method is not computationally much more demanding than the realisation of any ordinary ($L = 1$) distribution. In the case of multicomponent signals, this method will produce sigal power concentrated at the resulting instantaneous frequency, according to theorem 1 [14].

5.2 Recursive method: S-reduced interference distributions

Although the Wigner distribution itself satisfies most of the desired properties in the time–frequency representation of a signal, it is rarely used in its original form. The main reason lies in the very emphatic crossterm effects. These effects may be even more emphasised in the L-class distribution for $L > 1$, since the Lth power of signal may increase the number of cross-terms [10, 11]. Unfortunately, some of these terms behave as the regular autoterms, [13]. Thus, the straightforward generalisation of the RID distributions (Choi–Williams, Zao–Atlas–Marks, Born–Jordan, Sinc, ... [1, 2]) would reduce ouly a limited number of crossterms resulting from the product of $x^L(t + \tau/2L)$ and $x^{*L}(t-\tau/2L)$. For the L-distributions, we have achieved reduction (or complete removal) of all crossterms using the recursive S-method, as well as avoiding signal oversampling [6, 7, 10, 19, 20]. This method, although very efficient in the realisation of the L-class of distributions, if applied in a straightforward manner on the S-class would produce qualitatively the same result as the direct realisation, i.e. the obtained distribution would be completely concentrated at the signal's resulting instantaneous frequency [14]. However, we will present here its modified version which will be efficient in the realisation of the S-class of distributions.

The basic idea is very simple. If the signal does not have unity amplitude, then instead of convolving $SC_{L/2}(t, \omega + \lambda)$ and $SC_{L/2}(t, \omega - \lambda)$ in eqn. 16, we will convolve $SC_{L/2}(t, \omega + \lambda)$ and $SC^{(n)}_{L/2}(t, \omega - \lambda)$. Superscript $^{(n)}$ denotes a normalised version of distribution $SC_{L/2}(t, \omega)$, i.e. distribution $SC_{L/2}(t, \omega)$ if all signal components had unity amplitude. In this way, we get a distribution of the order of L with amplitude of the order of $L/2$, i.e. we have resolved the problem of how not to increase the order of amplitude during the recursions. Starting from the distribution that is crossterms-free, we may control (reduce or remove) the crossterms in the subsequent iterations using function $P(\lambda)$ in eqn. 16 (which is of a lowpass filter type [7, 10]), while the order of signal amplitude is kept uncharged. The modified form of eqn. 16 is

$$SCM_L(t,\omega) = \int_\lambda P(\lambda) SC_{L/2}(t,\omega+\lambda) SC^{(n)}_{L/2}(t,\omega-\lambda)\frac{d\lambda}{\pi} \tag{19}$$

Here, we will provide some additional details on the realisation of eqn. 19. Consider a multicomponent signal

$$x(t) = \sum_{i=1}^{P} x_i(t)$$

Our aim is to obtain a distribution such that, theoretically, it is equal to a sum of the distributions of each component separately, i.e.

$$SC_{L,x}(t,\omega) = \sum_{i=1}^{P} SC_{L,x_i}(t,\omega)$$

The marginal properties, in this case, are

$$\frac{1}{2\pi}\int_\omega SC_{L,x}(t,\omega)d\omega = \sum_{i=1}^{P} |x_i(t)|^2$$

and

$$\int_t SC_{L,x}(t,\omega)dt = \sum_{i=1}^{P} |X_i(\omega)|^2 \tag{20}$$

Let us start from the short time Fourier transform of $x(t)$

$$\begin{aligned} STFT(t,\omega) &= \int_\tau w(\tau)x(t+\tau)e^{-j\omega\tau}d\tau \\ &= \int_\tau w(\tau)A(t+\tau)e^{j\phi(t+\tau)}e^{-j\omega\tau}d\tau \end{aligned} \tag{21}$$

As it is known, this transform does not have cross-terms, in the time–frequency plane, between separated signal components. To produce higher-order distributions we will need an amplitude normalised version of $STFT(t, \omega)$ which will be denoted by $STFT^{(n)}(t, \omega)$ and defined as

$$STFT^{(n)}(t,\omega) = \int_\tau w(\tau)e^{j\phi(t+\tau)}e^{-j\omega\tau}d\tau$$

If amplitude $A(t)$ is slow-varying, we may easily get $STFT^{(n)}(t, \omega)$ from $STFT(t, \omega)$ as

$$STFT^{(n)}(t,\omega) = STFT(t,\omega)\sqrt{\frac{E_w}{E_x(t)}} \tag{22}$$

where $A(t) \cong \sqrt{[E_x(t)E_w]}$ and $E_x(t) = 1/2\pi \int_\omega |STFT(t, \omega)|^2 d\omega$. In the derivation of the above equation Parseval's theorem is used ($1/2\pi \int_\omega |STFT(t, \omega)|^2 d\omega = \int_\tau |w(\tau)A(t + \tau)|^2 d\tau$). The slow-varying amplitude $A(t)$ means that $w(\tau)A(t + \tau) \approx w(\tau)A(t)$. This condition may be written in a less restrictive form. Assume, for example a Hanning window $w(\tau)$ and $A(t + \tau) = A(t) + A'(t)\tau + A''(t)\tau^2/2$. The scaling factor in eqn. 22 remains the same if $A^2(t) >> [A'^2(t) + A(t)A''(t)]/6.17 + A''^2(t)/120$, i.e. if $A(t)$, $A'(t)$, $A''(t)$ are of the same order. In the example, we will see that the results will not be significantly degraded even if this condition is not satisfied.

If the signal is multicomponent, with slow-varying amplitudes of each component, as well as with components separated along the frequency axis for any t (i.e. signal components lie, along ω, inside regions Ω_i, which do not overlap), then

$$STFT^{(n)}(t,\omega) = \sum_{i=1}^{P} STFT(t,\omega)\sqrt{\frac{E_w}{E_{x_i}(t)}}\Pi_{\Omega_i}(\omega) \quad (23)$$

where $\Pi_{\Omega_i}(\omega)$ is equal to unity for ω inside Ω_i and zero outside. In the numerical realisation, the values of $STFT^{(n)}(t, \omega)$ outside Ω_i were not assumed to be zero, but they are left unchanged with respect to their original values, producing the spectrogram as a final result outside Ω_i.

Knowing $STFT(t, \omega)$ and $STFT^{(n)}(t, \omega)$, we may easily realise distribution

$$S_1(t,\omega) = \int_{\tau} w^2\left(\frac{\tau}{2}\right) A\left(t+\frac{\tau}{2}\right) e^{j\phi(t+\frac{\tau}{2})} e^{-j\phi(t-\frac{\tau}{2})} e^{-j\omega\tau} d\tau$$

according to the S-method, as

$$S_1(t,\omega) = \frac{1}{\pi}\int_{\theta} P(\theta) STFT(t,\omega+\theta) STFT^{*(n)}(t,\omega-\theta) d\theta \quad (24)$$

where $P(\theta)$ is a frequency domain window function, which has to be wide enough to ensure the integration over autoterms and narrow enough to avoid crossterms [7, 10]. Recently, we proposed a very simple signal-dependent and self-adaptive window $P(\theta)$ [19]. After we get crossterms free $S_1(t, \omega)$, then we may get the S-distribution for $L = 2$

$$SD_2(t,\omega) = \int_{\tau} w^4\left(\frac{\tau}{4}\right) A\left(t+\frac{\tau}{4}\right) A\left(t-\frac{\tau}{4}\right) \times e^{j2\phi(t+\frac{\tau}{4})} e^{-j2\phi(t-\frac{\tau}{4})} e^{-j\omega\tau} d\tau$$

convolving two $S_1(t, \omega)$ as

$$SD_2(t,\omega) = \frac{1}{\pi}\int_{\theta} P(\theta) S_1(t,\omega+\theta) S_1(t,\omega-\theta) d\theta \quad (25)$$

where again $P(\theta)$, eliminates (reduces) crossterms, while the autoterms are the same as in the original S-distribution of the order of two. This procedure may be continued up to any order of the S-distribution. Namely, convolving $SD_2(t, \omega)$ and its normalised version $SD_2^{(n)}(t, \omega)$, we get $SD_4(t, \omega)$, and so on. Efficiency of the proposed realisation (as well as some other details on the realisation itself) will be demonstrated, in the next Section, using a very complex numerical example, including signal components which do not fully comply with the described conditions.

6 Example

As an example, consider a real multicomponent signal

$$x(t) = e^{-4(t-0.5)^2}\cos[180\pi(t-0.5)^3 + 50\pi t] + 0.5e^{-4(t-0.5)^2}\cos(20\pi t) + 0.707e^{-40(t-0.5)^2}\cos(50\pi t^2 + 150\pi t) \quad (26)$$

within the interval [0, 1]. For the numerical realisation we have used a Hanning window $w(\tau)$ of unity width with $N = 256$ samples, as well as a rectangular window $P(\theta)$ with signal-dependent width. The realisation is done according to the procedure described in Section 5.2. Here, we will provide some additional details.

(*a*) First we have to determine regions Ω_i, to obtain the scaling factors in eqn. 23. For this purpose we assume reference level $R_{lev}(t)$ for a given instant t, as $R_{lev}(t) = \max_\omega\{|STFT(t, \omega)|^2\}/Q^2$. Regions Ω_i are defined by the compact regions where $|STFT(t, \omega)|^2 > R_{lev}(t)$. Factor Q defines reference level. For non-noisy signals its value may be very high. But, in the noisy cases, to avoid false autoterm detection, this factor should not be too large. We found a very appropriate value for non-noisy, as well as for noisy, signals $Q^2 = 25$ (or the values around this, for example, from $Q^2 = 10$ to $Q^2 = 50$, since the quality of a distribution presentation is not too sensitive to this parameter). Region Ω_i starts at the first ω where $|STFT(t, \omega)|^2 > R_{lev}, (t)$, but to avoid the break of region Ω_i in the noisy cases, as well as in the cases when the amplitude of a single autoterm changes sign, we assumed that Ω_i ends not if a single value of $|STFT(t, \omega)|^2$ is below $R_{lev}(t)$, but if three subsequent values of $|STFT(t, \omega)|^2$ (at $(k-1)\Delta\omega$, $k\Delta\omega$ and $(k+1)\Delta\omega$) are less than the reference level.

(*b*) After we have found regions Ω_i, then the scaling factors for each region, according to eqn. 23, are determined.

(*c*) Convolution of $STFT(t, \omega)$ and its normalised version $STFT^{(n)}(t, \omega)$ is calculated according to eqn. 24. Here, we have used the signal-dependent rectangular window $P(\theta)$ width. For a given ω inside Ω_i, integration over θ (determined by the width of $P(\theta)$) is performed until any $\omega + \theta$ or $\omega - \theta$ is outside Ω_i. In this way we completely avoid the possibility of crossterms between nonoverlapping autoterms. Also, the accumulation of noise is kept at the lowest possible level, avoiding all summations outside an autoterm, [19].

(*d*) Finally, convolving two $S_1(t, w)$, according to eqn. 25, we get crossterm-free S-distribution of the order of two. Since a high autoterm concentration is achieved in $S_1(t, \omega)$, then a very narrow window $P(\theta)$ in eqn. 25 may be used. Even with $P(\theta) = \pi\delta(\theta)$ we get very good results for all considered signals. Thus, in this step, this window form is assumed in all examples.

(*e*) If one wants to get the S-distribution of a higher order than $L = 2$ (corresponding to fourth-order distributions) then the steps from *a* to *d* have to be repeated starting from $SD_2(t, \omega)$ instead of $STFT(t, \omega)$, and so on, for $L = 4, 8, \ldots$

The Wigner distribution of the analytic part of a signal eqn. 26 is presented in Fig. 1*a*. The analytic part is used to avoid crossterms between positive and negative frequency components. The S-method (autoterms almost as in the Wigner distribution, but without crossterms, [7, 10, 19]) is presented in Fig. 1*b*. Here, we did not use the analytic part of the signal, since the crossterms between components of positive and negative frequencies are eliminated in the same way as the other crossterms. Crossterm free S-distribution, with $L = 2$, is shown in Fig. 1*c*. Further improvement of the concentration may be achieved using the distributions with $L = 4$ and $L = 8$ (Fig. 1*d*). To illustrate the distributions' convergence we provided logarithmic scale plots at instant $t = 0.5$, in Figs. 2–5. Note that in all examples, the sampling interval is taken according to the sampling theorem. The reference level factor is $Q^2 = 50$. For some additional examples, including noisy signals as well as signals whose components intersect, we refer the reader to [14, 15]. In the noisy cases the reference

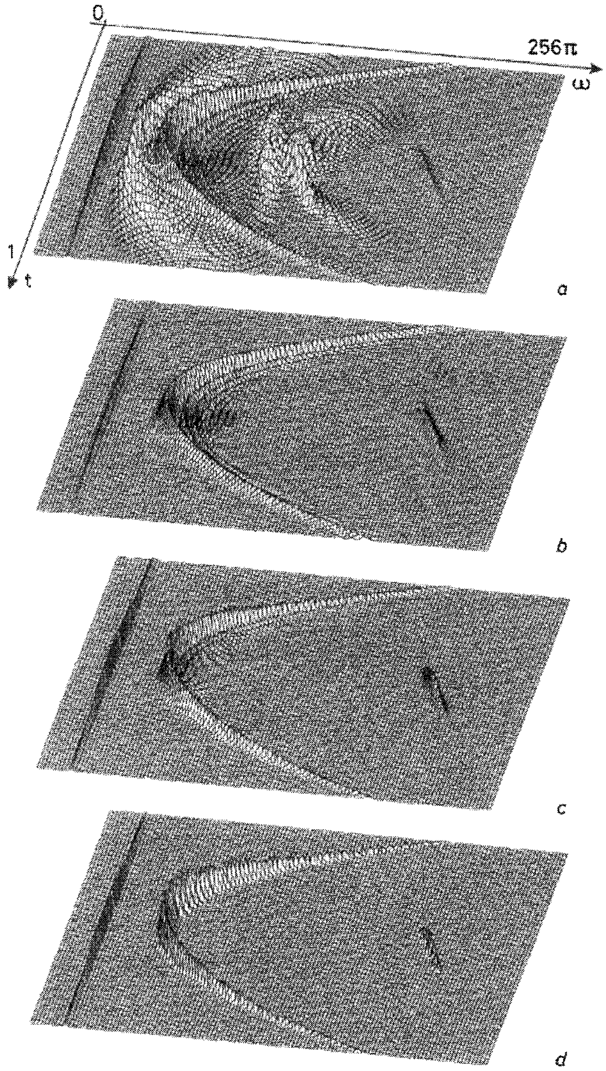

Fig. 1 *Time–frequency representation of multicomponent signal*
a Wigner distribution of signal's analytic part
b S-distribution with $L = 1$ (S-method)
c Crossterms-free S-distribution with $L = 2$
d S-distribution with $L = 8$

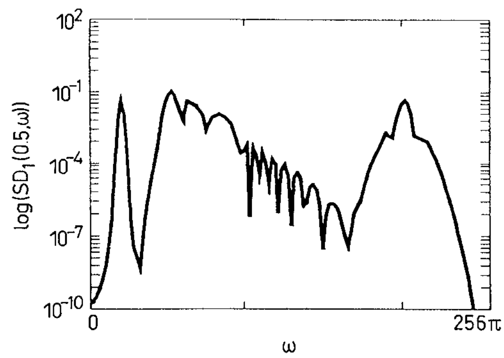

Fig. 2 *S-distribution, with $L = 1$, at $t = 0.5$*

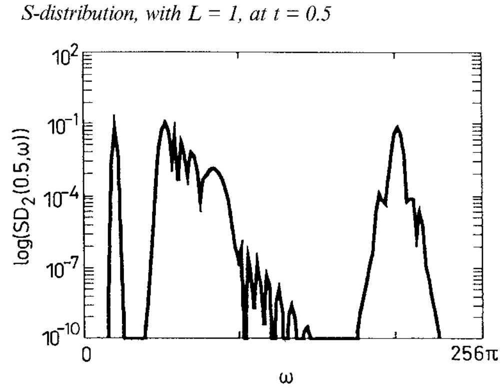

Fig. 3 *S-distribution, with $L = 2$, at $t = 0.5$*

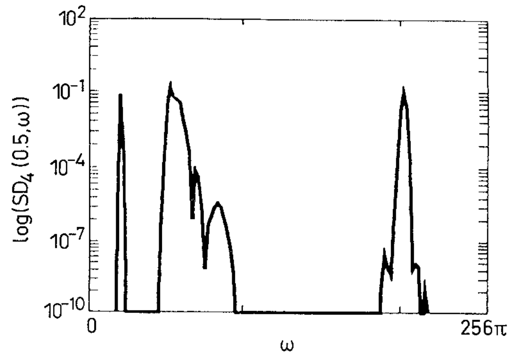

Fig. 4 *S-distribution, with $L = 4$, at $t = 0.5$*

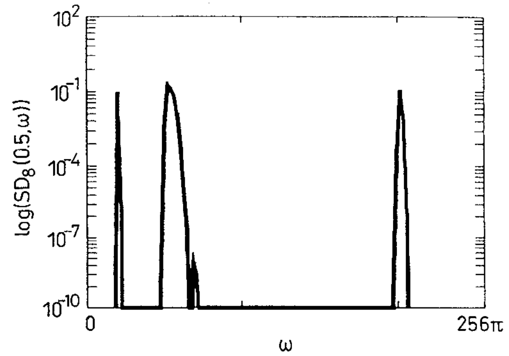

Fig. 5 *S-distribution, with $L = 8$, at $t = 0.5$*

```
clear              %PROGRAM
global Ew
N=256; Q=sqrt(50); Ew=sum((hanning(N)).^2)/N;
t=-0.5:1/N:1.5; y=sigsd(t); n=0;
for k=1:4:(length(y)-N); k, n=n+1;
STFT=fftshift(fft(y(k:k+N-1).*(hanning(N))'))/N;
STFT=STFT(N/2+1:N);
SPEC=[SPEC;abs(STFT).^2];
SD1(n,:)=css(STFT,Q,0);
W=css(STFT,Q,1); SD2(n,:)=css(W,1,0);
SD4(n,:)=css(SD2(n,:),Q,1);
SD8(n,:)=css(SD4(n,:),Q,1);
end, mesh(SD1,[10 80]), pause, mesh(SD4,[10 80])
```

```
function W=css(S,Q,U)
% S-input sequence; Q-level factor
% For normalization U=1, otherwise U=0
N=length(S); R=max(abs(S))/Q; b=0; E=zeros(1,N);
for k=3:N-1
  if (abs(S(k))>R)|(max(abs(S(k-1:k+1)))>R & b~=0)
    b=b+1; E(k)=E(k-1)+abs(S(k))^2;
    else, E(k-b:k-2)=E(k-2)*ones(1,b-1); b=0;
end, end
F=sqrt(E/Ew);
 for k=1:N, if (F(k)==0 | U==0), F(k)=1; end, end,
for k=1:N
W(k)=(abs(S(k)).^2)/F(k)*2; SmC=1; i=0;
  while (SmC==1 & i<N), i=i+1; p=k+i; m=k-i;
      if (m>0 & p<N), if (E(p)>0 & E(m)>0)
      W(k)=W(k)+2*real(S(p)*conj(S(m)))/F(m)*2;
      else, SmC=0; end, else, SmC=0; end
end, end, end
```

```
function y=sigsd(t);
y=exp(-4*(t-.5).^2).*cos(180*pi*(t-.5).^3+50*pi*t);
end
```

Fig. 6 *MATLAB program (including function files) for the S-distributions realisation*

factor could have smaller values, for example $Q^2 = 16$ or $Q^2 = 25$. A very short and self-contained MATLAB program, according to the above algorithm is presented in Fig. 6.

7 Conclusions

The S-class of distribution, as a generalisation of the S-distribution, is proposed. A method for the efficient realisation of the S-class of distributions is presented.

Theory is illustrated using a numerical example. The proposed distributions may achieve arbitrary high concentration at the instantaneous frequency, satisfying the marginal properties. Out of the known distributions, this was possible only in the very special case of the linear frequency modulated signal using the Wigner distribution.

8 Acknowledgment

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