Time–Frequency Distributions With Complex Argument

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***Abstract*—A distribution highly concentrated along the group delay or the instantaneous frequency (IF) is presented. It has been defined by introducing signal with a complex argument in time–frequency (TF) analysis. Realization of a signal with complex argument, using signal with a real argument, is described. The reduced interference realization of the complex argument distribution, in the case of multicomponent signals, is presented. The proposed distribution has been used for the IF estimation. It has been shown that the estimation could be improved with respect to the Wigner distribution based one since the bias could be significantly reduced with only a slight increase of the variance. The theory is illustrated by examples.**

***Index Terms*—Estimation, instantaneous frequency, nonstationary signal analysis, signal analysis, spectral analysis, time–frequency analysis, Wigner distribution.**

I. Introduction

TIME–FREQUENCY (TF) analysis deals with signal representations in the joint TF plane. The simplest TF representation (the short-time Fourier transform) is a straightforward extension of the Fourier transform obtained by introducing a lag window that localizes the spectral content around the considered time instant. The energetic version of this transform is called spectrogram. It belongs to the quadratic TF representations. In order to improve concentration in the TF plane, other quadratic distributions are introduced [11]. The Wigner distribution (WD) originally defined in quantum mechanics is commonly used as a basic (and the best concentrated signal independent [12]) quadratic TF representation. In order to reduce undesired nonlinearity effects, manifesting themselves as the cross-terms, various reduced interference distributions have been introduced [11].

When the instantaneous frequency (IF) is a nonlinear function of time, concentration of the signal's TF representation could be improved by using higher order multidimensional time-varying spectra, e.g., the Wigner higher order spectra, and its dual form—the multitime Wigner distributions [14], [31]. Other approaches are based on the specific representations for the assumed signal forms [2]–[6], [15], [16], [18], [22]–[24]. Interesting higher order representations, from the practical realization standpoint, are those that can be reduced to the two-dimensional (2-D) TF plane. This reduction can be achieved either by projections or by slicing, resulting in, for example, the L-Wigner distributions or the polynomial Wigner–Ville distributions [9], [10], [25], [30], [31].

Based on the initial idea from [28], in this paper, we introduce and develop the complex argument distributions. They are interesting from the *theoretical point of view* since they use complex frequency argument (in the Laplace domain) and a corresponding *complex-lag argument* in the time domain. These forms are able to produce almost completely concentrated representations along the group delay or the IF. A method for the realization of the signal with complex-valued argument, based on the signal with real-valued argument, is presented. It uses the relation between the Fourier transform and the Laplace transform and the analytic extension (continuation) of the signal [20]. A procedure for application of this distribution on the TF analysis of multicomponent signals is presented. It may produce cross-terms reduced (or cross-terms free) forms of the complex argument distribution. This procedure is also efficient in reducing the noise influence on the TF representation of monocomponent noisy signals. The complex argument distribution, as an IF estimator in the noisy signal cases, is analyzed. The estimator's variance and bias are derived.

The paper is organized as follows. A TF representation highly concentrated along the group delay is introduced in the next section. Its definition is based on the complex-frequency and the Laplace transforms. Properties and various forms of this representation are given. Since the IF is practically more important parameter than the group delay, dual representation to the former is defined next. A notion dual to the complex-frequency referred to as the "complex-time" or complex-lag argument is used for this purpose. Since the signal is available along the real time axis only, the tools for calculation of a complex-valued argument form of the signal are considered and proposed. Properties of the complex argument TF representation are studied at the end of this section. Section III gives the discrete forms of the proposed representations, including the one that could be considered to be a corrected form of the scaled WD. This interpretation is convenient for numerical realizations. Realization of the proposed distributions for multicomponent signals is studied in Section IV. Influence of noise to the IF estimation, based on the complex argument TF representation, is the topic of Section V and the Appendix. The theory is illustrated on examples with monocomponent signals and IF estimation as well as on the TF representation of multicomponent signals.

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II. Complex Argument Distribution: Definitions and Properties

A. Complex-Frequency Distribution

The Fourier transform pair of a signal $x(t)$ is defined by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega. \tag{1}$$

The WD may be written in terms of either the signal itself or its Fourier transform

$$WD(t,\omega) = \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j\omega\tau}\, d\tau$$
$$WD(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\omega + j\,\frac{\theta}{2}\right) X^*\left(j\omega - j\,\frac{\theta}{2}\right) e^{j\theta t}\, d\theta. \tag{2}$$

In the analysis that follows, we will first use the WD expressed in terms of the signal's Fourier transform since it will be conceptually much easier to introduce a complex argument in this domain.

Consider a signal whose Fourier transform is of the form

$$X(j\omega) = A(\omega) e^{j\varphi(\omega)} \tag{3}$$

where $A(\omega)$ is a slow varying function with respect to $\varphi(\omega)$. For these kinds of signals

$$\begin{aligned} X\left(j\omega + j\,\frac{\theta}{2}\right) X^*\left(j\omega - j\,\frac{\theta}{2}\right) &\cong A^2(\omega) e^{j(\varphi(\omega+\theta/2)-\varphi(\omega-\theta/2))} \\ &\cong A^2(\omega) e^{j(\varphi'(\omega)\theta + 2\varphi^{(3)}(\omega)\theta^3/(2^3 3!) + 2\varphi^{(5)}(\omega)\theta^5/(2^5 5!)+\cdots)} \end{aligned} \tag{4}$$

where $\varphi(\omega \pm \theta/2)$ has been expanded into the Taylor series around ω. The WD is concentrated along the group delay $t_g(\omega) = -\varphi'(\omega)$ with a spread function depending on the higher (third, fifth, …) order derivatives of $\varphi(\omega)$, i.e.,

$$\begin{aligned} WD(t,\omega) &\cong A^2(\omega)\delta(t+\varphi'(\omega)) *_t \\ & IFT_\theta \left\{ e^{j(\varphi^{(3)}(\omega)\theta^3/(2^2 3!) + \varphi^{(5)}(\omega)\theta^5/(2^4 5!)+\cdots)} \right\} \end{aligned} \tag{5}$$

where IFT denotes the inverse Fourier transform. For $\varphi^{(3)}(\omega) = \varphi^{(5)}(\omega) = \cdots = 0$ and $A(\omega) = A$, the WD is completely concentrated along the group delay $WD(t,\omega) = A^2\delta(t+\varphi'(\omega))$.

When the higher order derivatives exist, then the concentration can be improved by introducing a distribution with the complex argument. In the frequency domain, the notion of complex frequency is well established and studied in signals and systems, within the Laplace transform framework [17].

Definition 1: The complex argument distribution is defined as

$$\begin{aligned} CD(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} & X\left(j\omega + j\,\frac{\theta}{4}\right) X^*\left(j\omega - j\,\frac{\theta}{4}\right) \\ & \times X^j\left(j\omega + \frac{\theta}{4}\right) X^{-j}\left(j\omega - \frac{\theta}{4}\right) e^{j\theta t}\, d\theta \end{aligned} \tag{6}$$

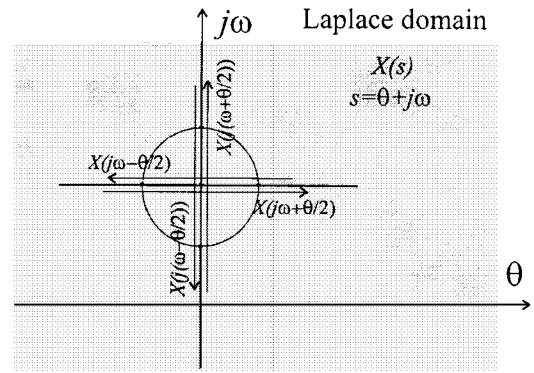

Fig. 1. Complex argument distribution illustration in the Laplace transform domain.

where $X(j\omega \pm \theta/4)$ is the Laplace transform of $x(t)$ at $s = \pm\theta/4 + j\omega$

$$X\left(j\omega \pm \frac{\theta}{4}\right) = \int_{-\infty}^{\infty} x(t) e^{\mp\theta t/4} e^{-j\omega t}\, dt. \tag{7}$$

It converges for $\int_{-\infty}^{\infty} |x(t)\exp(\mp\theta t/4)| dt < \infty$. Integration in (6) is performed over the entire complex argument plane, i.e., for all ω and θ; see Fig. 1. Note that the Laplace transform $X(j\omega \pm \theta/4)$ may not exist for some θ even if the associated Fourier transform exists [17]. The convergence of (7) is guaranteed for any finite θ for absolute values-limited $|x(t)| < \infty$ and time-limited signals $x(t) = 0$ for $|t| > t_m$.[1] The problem may appear in numerical calculation since (7), although being finite, can assume large values for some θ. This is the reason why a special attention will be paid to the discrete formulation and calculation of the complex-argument terms in (6).

A complex-valued number $z = re^{j\alpha}$ may easily be raised to the jth power according to $z^j = e^{j\ln(r)}e^{-\alpha}$. A procedure for raising the signal's Laplace transform to the jth power, avoiding problems that may occur due to the specific form of the complex function $f(z) = z^j$, will be discussed later.

Property: For time-limited signals $x(t) = 0$ for $|t| > t_m$, we have

$$X\left(j\omega - \frac{\theta}{4}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} X(j\lambda)\, \frac{\sinh\{[\theta/4 + j(\lambda-\omega)]t_m\}}{\theta/4 + j(\lambda-\omega)}\, d\lambda. \tag{8}$$

Proof: For the assumed time-limited signals, by substituting signal $x(t)$ from (1) into (7), we get (8). It converges for any finite θ and t_m if $\int_{-\infty}^{\infty} |X(j\lambda)|\, d\lambda < \infty$.

Expression (8) can be understood as a relation between the Fourier transform $X(j\lambda)$ and the Laplace transform $X(s)$, $s = j\omega - \theta/4$ for time-limited signals.

Property: The complex argument distribution of a signal $X(j\omega) = Ae^{j\varphi(\omega)}$ is concentrated along the group delay

[1] For the analysis of signals with a finite bandwidth and infinite duration in time, the "complex-time" distribution could be used. It is introduced in Section II-B as a tool for the instantaneous frequency estimation. Signals with infinite duration, in both the time and frequency domain, can be analyzed by the complex argument distributions using a general mathematical concept of analytic extension [see the text after (20)in Section II-B].

$t_g(\omega) = -\varphi'(\omega)$ with the lowest spreading term being of the fifth order, i.e.,

$$CD(t,\,\omega) = A^2\delta(t+\varphi'(\omega)) *_t \\ IFT_\theta\left\{e^{j(\varphi^{(5)}(\omega)\theta^5/(4^4 5!)+\varphi^{(9)}(\omega)\theta^9/(4^8 9!)+\cdots)}\right\}. \quad (9)$$

Proof: Expansion of the complex argument function $\varphi(\omega \pm j\theta/4)$ into a complex-valued Taylor series results in

$$X^j\left(j\omega+\frac{\theta}{4}\right)X^{-j}\left(j\omega-\frac{\theta}{4}\right) \\ \cong e^{\varphi(\omega+j\theta/4)-\varphi(\omega-j\theta/4)} \\ = e^{j(\varphi'(\omega)\theta/2-2\varphi^{(3)}(\omega)\theta^3/(4^3 3!)+2\varphi^{(5)}(\omega)\theta^5/(4^5 5!)+\cdots)}. \quad (10)$$

By substituting this expression and (4) and (5) (with $\theta/4$ instead of $\theta/2$) into $CD(t,\,\omega)$, we get (9).

Relation (9) means that $CD(t,\,\omega)$ is a distribution with amplitude A^2 concentrated along the group delay with the lowest disturbing term depending on the fifth derivative of the phase function divided by a factor of $4^4 5! = 30\,545 \sim 10^4$. Therefore, we will get a completely concentrated distribution for the phase of up to the fourth-order polynomial function of time. The first disturbing term is of the fifth order. It is divided by 4^4, as compared with the same term in the spectrogram, or by 2^4 with respect to the same term in the WD. The next disturbing term is of the ninth order $\varphi^{(9)}(\omega)\theta^9/(4^8 9!) \to 0$. Therefore, we may expect a highly (almost ideally) concentrated distribution, along the group delay, for any signal whose Fourier transform is of the form $X(j\omega) = Ae^{j\varphi(\omega)}$.

Numerical realization of complex frequency distribution can be simplified by using the following.

Property: For signals of the form $X(j\omega) = Ae^{j\varphi(\omega)}$, it holds that

$$X^j\left(j\omega+\frac{\theta}{4}\right)X^{-j}\left(j\omega-\frac{\theta}{4}\right) = e^{j\ln|X(j\omega+\theta/4)/X(j\omega-\theta/4)|}. \quad (11)$$

Proof: Consider

$$X\left(j\omega \pm \frac{\theta}{4}\right) \\ = Ae^{j\varphi(\omega\mp j\theta/4))} \\ = Ae^{j\varphi(\omega)\pm\varphi'(\omega)\theta/4-j\varphi^{(2)}(\omega)\theta^2/(4^2 2!)\mp\varphi^{(3)}(\omega)\theta^3/(4^3 3!)+\cdots}. \quad (12)$$

It may be concluded that

$$\frac{X\left(j\omega+\dfrac{\theta}{4}\right)}{X\left(j\omega-\dfrac{\theta}{4}\right)} = e^{2\varphi'(\omega)\theta/4-2\varphi^{(3)}(\omega)\theta^3/(4^3 3!)+\cdots}$$

resulting in

$$e^{j\ln|X(j\omega+\theta/4)/X(j\omega-\theta/4)|} = e^{j\varphi'(\omega)\theta/2-j2\varphi^{(3)}(\omega)\theta^3/(4^3 3!)+\cdots} \quad (13)$$

producing exactly the term $C(\omega,\,\theta) = X^j(j\omega+\theta/4)X^{-j}(j\omega-\theta/4)$, which is given by (10). This completes the proof of (11). Note that the ratio $X(j\omega+(\theta/4))/X(j\omega-(\theta/4))$ is real-valued, thus eliminating a possible phase ambiguity in calculating its jth power.

Note: Based on the relation between $X(j\omega-(\theta/4))$ and $X(j\omega+(\theta/4))$

$$\left|X\left(j\omega-\frac{\theta}{4}\right)\right| = \left|\frac{A^2}{X\left(j\omega+\dfrac{\theta}{4}\right)}\right| \quad (14)$$

which follows from (12), we can use only one of these terms in calculation of (13).

Now, the complex-argument distribution can be written as

$$CD(t,\,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\omega+j\,\frac{\theta}{4}\right)X^*\left(j\omega-j\,\frac{\theta}{4}\right) \\ \times C(\omega,\,\theta)e^{j\theta t}\,d\theta. \quad (15)$$

This form reduces the complexity and also gives the possibility of avoiding the problem of cross-terms when one generalizes the numerical approach to the multicomponent signals. Based on (15), the complex argument distribution can be related to the general quadratic class of distributions [11].

B. Complex-Lag Distribution

It has been shown that the concentration along the *group delay* can be improved by using the complex-frequency argument in the TF representation. In practice, the IF is more commonly used signal parameter than the group delay. TF representation producing improved concentration along the IF can be introduced by replacing frequency with time since any definition in frequency domain can be reintroduced in its dual form. In order to define a representation with complex-valued argument, we have to introduce the quantity that will be related to the time axis in the same way as the complex frequency is related to the frequency axis. This mathematical quantity will be referred to as the "complex time" or complex-lag argument.

For an FM signal

$$x(t) = re^{j\phi(t)} \quad (16)$$

TF representations may be written as

$$TFR(t,\omega) = 2\pi r^{2q}\delta(\omega-\phi'(t)) *_\omega W(\omega) *_\omega FT\left\{e^{jQ(t,\tau)}\right\} \quad (17)$$

where

$Q(t,\,\tau)$ factor causing distribution spread around the IF;
$W(\omega)$ lag-window's Fourier transform;
q constant.

According to the analysis in the Fourier domain, we can conclude that a significant improvement in concentration, along the IF, can be achieved by defining a distribution with the complex-lag argument [28], which is dual to (6).

TABLE I
SPREAD FACTOR $Q(t, \tau)$ IN SOME TIME-FREQUENCY DISTRIBUTIONS

Distribution	Definition Spread factor
STFT	$STFT(t,\omega)=\int_{-\infty}^{\infty} w(\tau)x(t+\tau)e^{-j\omega\tau}d\tau$ $Q(t,\tau)=\phi^{(2)}(t)\frac{\tau^2}{2!}+\phi^{(3)}(t)\frac{\tau^3}{3!}+\phi^{(4)}(t)\frac{\tau^4}{4!}+\ldots$
Wigner distribution	$WD(t,\omega)=\int_{-\infty}^{\infty} w(\tau)x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j\omega\tau}d\tau$ $Q(t,\tau)=\phi^{(3)}(t)\frac{\tau^3}{2^2 3!}+\phi^{(5)}(t)\frac{\tau^5}{2^4 5!}+\ldots$
L-Wigner distribution	$LWD(t,\omega)=\int_{-\infty}^{\infty} w(\tau)x^L(t+\frac{\tau}{2L})x^{*L}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau$ $Q(t,\tau)=\phi^{(3)}(t)\frac{\tau^3}{2^2 3! L^2}+\phi^{(5)}(t)\frac{\tau^5}{2^4 5! L^4}+\ldots$
Fourth order polynomial Wigner-Ville distribution	$PWD(t,\omega)=\int_{-\infty}^{\infty} w(\tau)x(t+0.675\tau)x^*(t-0.675\tau)$ $\times x^*(t+0.85\tau)x(t-0.85\tau)e^{-j\omega\tau}d\tau$ $Q(t,\tau)=-0.327\phi^{(5)}(t)\frac{\tau^5}{5!}-0.386\phi^{(7)}(t)\frac{\tau^7}{7!}+\ldots$

Definition 2: The complex-lag distribution is defined by

$$CTD(t, \omega)=\int_{-\infty}^{\infty} x\left(t+\frac{\tau}{4}\right)x^*\left(t-\frac{\tau}{4}\right) \times x^{-j}\left(t+j\frac{\tau}{4}\right)x^{j}\left(t-j\frac{\tau}{4}\right)e^{-j\omega\tau}\,d\tau. \tag{18}$$

The spread factor $Q(t, \tau)$ for this distribution, according to (9), is

$$Q(t, \tau)=\phi^{(5)}(t)\frac{\tau^5}{4^4 5!}+\phi^{(9)}(t)\frac{\tau^9}{4^8 9!}+\cdots. \tag{19}$$

The dominant term in $Q(t, \tau)$ is of the fifth order. All existing terms are significantly reduced as compared with the respective ones in the WD. Some distributions, which are interesting from the point of view of concentration along the IF, are presented in Table I. The artifacts in the L-Wigner distribution [30], [31], [34] are reduced with respect to the ones in the WD. By increasing the distribution order L, we can additionally improve its concentration. The polynomial Wigner–Ville distribution [7], [9], [10] produces complete concentration along the IF for the order adjusted to the signal, whereas it is sensitive to the fifth-order term (in this case a higher order polynomial Wigner distribution should be used). Comparison of the complex-lag distribution and the polynomial Wigner–Ville distribution, including illustrative examples, is performed in [32] as well. In order to improve the concentration property of the polynomial WV distribution, its complex-lag counterpart is introduced in [21].

Definition 3: The continuous form of the "complex-time" signal

$$x(\varsigma)=x(t+j\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-\omega\tau}e^{j\omega t}\,d\omega \tag{20}$$

is dual to the Laplace transform of $x(t)$. Complex time is denoted by $\varsigma=t+j\tau$.

The signal with a "complex-time" argument could be calculated in the same way as the Laplace transform is calculated from the Fourier transform. According to the analysis for the complex-frequency, it is easy to conclude that $x(\varsigma)$ converges within the entire complex plane ς if $x(t)$ is a bandlimited signal.

Signals with infinite duration in both time and frequency, in general, do not satisfy the convergence conditions either for (7) or for (20). Although these signals are not of practical interest (since they can not be sampled and processed numerically), they still can be treated analytically, within the framework of complex argument distributions, but with an appropriate formal mathematical interpretation of the complex argument function $x(\varsigma)=x(t+j\tau)$ as an analytic extension (continuation) of the real-valued argument function $x(t)$ [20]. This extension has already been implicitly used in the analysis of the distribution with complex frequency [for writing the expression of $X(j\omega+\theta/4)$ based on $X(j\omega)=Ae^{j\varphi(\omega)}$]. For example, an analytic extension of the signal $x(t)=\exp(jat^2)$ is $x(\varsigma)=\exp(ja\varsigma^2)=\exp(ja(t+j\tau)^2)$. By inserting this form into the definition of $CTD(t, \omega)$, we get the expected result $CTD(t, \omega)=2\pi\delta(\omega-2at)$.

Property: In analogy with (15), the complex-lag distribution (18) of a signal $x(t)=re^{j\phi(t)}$ can be considered to be a frequency-scaled and corrected WD, where the correcting factor is both signal and TF dependent.

Proof: The correcting term can be obtained by using time-domain form of relations (12)–(15), which reads

$$\begin{aligned} c(t, \tau) &= x^{-j}\left(t+j\frac{\tau}{4}\right)x^{j}\left(t-j\frac{\tau}{4}\right) \\ &= e^{j\ln|x(t-j(\tau/4))/x(t+j(\tau/4))|} \\ &= e^{j\phi'(t)\tau/2-j2\phi^{(3)}(t)\tau^3/(4^3 3!)+\cdots} \end{aligned} \tag{21}$$

and

$$\begin{aligned} CTD(t, \omega) &= \int_{-\infty}^{\infty} x\left(t+\frac{\tau}{4}\right)x^*\left(t-\frac{\tau}{4}\right)c(t, \tau)e^{-j\omega\tau}\,d\tau \\ &= 2WD(t, 2\omega) *_\omega FT_\tau\{c(t, \tau)\}. \end{aligned} \tag{22}$$

The complex-lag distribution satisfies some other important TF representation.

Properties:

1) The complex-lag distribution is real *for the frequency modulated signals* $x(t)=re^{j\phi(t)}$.

This property follows from

$$\begin{aligned} R^*(t,\tau) &= \left[x\left(t+\frac{\tau}{4}\right)x^*\left(t-\frac{\tau}{4}\right)x^{-j}\left(t+j\frac{\tau}{4}\right)x^{j}\left(t-j\frac{\tau}{4}\right)\right]^* \\ &= \left[r^2 e^{j\phi'(t)\tau+j\phi^{(5)}(t)(\tau^5/4^4 5!)+j\phi^{(9)}(t)(\tau^9/4^8 9!)+\cdots}\right]^* \\ &= R(t, -\tau). \end{aligned}$$

2) *For any signal*, the complex-lag distribution satisfies the time-marginal property

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} CTD(t,\omega)\,d\omega = |x(t)|^2 \tag{23}$$

since

$$\left(\frac{x\left(t-j\,\dfrac{\tau}{4}\right)}{x\left(t+j\,\dfrac{\tau}{4}\right)}\right)^{j}_{|\tau=0} = 1.$$

3) The unbiased energy condition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} CTD(t,\omega)\,dt\,d\omega = \int_{-\infty}^{\infty} |x(t)|^2\,dt = E_x \tag{24}$$

is satisfied *for any signal*. This property follows directly from Property 2.

4) The frequency-marginal property is satisfied, provided that the spread factor $Q(t,\tau)$ may be neglected.

5) If $CTD(t,\omega)$ is a distribution of $x(t)$, then $CTD(t-T,\omega)$ is the distribution of $x(t-T)$.

This property is evident from the definition of $CTD(t,\omega)$.

6) If $CTD(t,\omega)$ is the complex-lag distribution of $x(t)$, then $CTD(t,\omega-\omega_0)$ is the distribution of signal $x(t)\exp(j\omega_0 t)$.

For the signal $\exp(j\omega_0 t)$, we have $R(t,\tau) = \exp(j\omega_0(t+\tau/4) - j\omega_0(t-\tau/4) + \omega_0(t+j\tau/4) - \omega_0(t-j\tau/4)) = \exp(j\omega_0\tau)$, producing the above property.

Generalization of 6: For the signal $x(t)\exp(j\alpha(t))$, the complex-lag distribution is of the form $CTD(t,\omega-\alpha'(t))$ for $\alpha(t)$ such that $\alpha^{(4)}(t)\equiv 0$.

7) The complex-lag distribution of the scaled signal $\sqrt{|a|}x(at)$ is $CTD(at,\omega/a)$.

III. Discrete Forms of the Complex Argument Distributions

Theoretically, numerical realization of the complex argument distributions will be simple if we know the analytic expression for the signal. However, in practical realizations, the values of $x(n)$ are available as a set of data along the real axis only. *The values of signal with complex argument are not known. They must be determined from the values on the real-time axis*. This problem can be solved in the same way as the Laplace transform is calculated from the Fourier transform. Note that it is a mathematically well-studied problem known as an analytical extension (continuation) of the real argument function [20].

A. z-Transform Based Discrete Form of the Complex Frequency Distribution

Consider a discrete-time signal $x(n)$ of a finite duration N. The discrete Fourier transform pair reads

$$X(k) = \sum_{n=-N/2}^{N/2-1} x(n)e^{-j2\pi nk/N}$$
$$x(n) = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} X(k)e^{j2\pi nk/N}. \tag{25}$$

The values of complex argument discrete Fourier transform are the samples of the z-transform $X(z)$ (see Fig. 2)

$$X(j\omega+\theta) \to X(z)_{|z=\exp(2\pi(jp+q)/N)} =$$
$$X(p-jq) = \sum_{n=-N/2}^{N/2-1} x(n)e^{-j2\pi n(p-jq)/N}. \tag{26}$$

Therefore, the discrete complex frequency distribution can be written as

$$CD(n,k) = \frac{4}{N}\sum_{q=-N/2}^{N/2-1} X(k+q)X^*(k-q) \times X^{j}(k-jq)X^{-j}(k+jq)e^{j(2\pi/N)4nq}. \tag{27}$$

B. Discrete Complex-Lag Forms

According to (18), the discrete pseudo form of the complex-lag distribution is given by

$$CTD(n,k) = \sum_{m=-N/2}^{N/2-1} w(m)x(n+m)x^*(n-m) \times x^{-j}(n+jm)x^{j}(n-jm)e^{-j(2\pi/N)4mk} \tag{28}$$

where $w(m)$ is a lag window. A constant factor of 4 is omitted in (28). Formally, the interpretation of $x(n\pm jm)$ can be done in the same way as in (26). However, since the complex argument in time is not so common as the complex frequency, we will also introduce an interpretation of $x(n\pm jm)$ in terms of the analytic signal extension. It is known that the analytical extension of function e^{jan} is $e^{ja\eta}$, where $\eta = n+jm$ is a complex argument. This analytical extension is valid for $|\eta| < \infty$. Consequently, the following definition can be given.

Definition 4: An analytic extension of the signal $x(n)$ is defined as a sum of the analytic extensions of complex exponential functions. It is of the form

$$x(\eta) = x(n+jm) = \frac{1}{N}\sum_{k=-N/2}^{N/2-1} X(k)e^{-(2\pi/N)mk}e^{j(2\pi/N)nk} \tag{29}$$

corresponding to (7), (20), or (25) with the region of convergence $|n+jm| < \infty$ for a finite N.

This is just a discrete dual form of (7). Expression (29) can now be used for an efficient numerical realization [28].

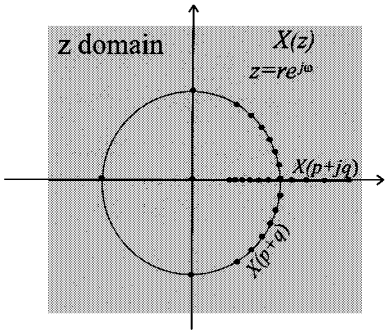

Fig. 2. Illustration of complex frequency distribution in the z domain for $p = 0$.

If we multiply $X(k) = FFT[x(n)]$ by $\exp(-2\pi mk/N)$, for a given m, then $x(n + jm)$ is obtained as $x(n + jm) = IFFT[X(k)\exp(-2\pi mk/N)]$.

The presented form of $x(n + jm)$ could directly be used for the realization of a complex-lag distribution. Real exponential functions $\exp(-2\pi mk/N)$, for large $-mk$, may be out of the computer precision range, significantly worsening the results. Thus, one should carefully use the above relations in the direct numerical realization.

Note that (29) is a dual z-transform, where the Fourier transform samples $X_N(k) = X(k)/N$ play the role of a signal, and

$$x(\mathfrak{z}) = \sum_{k=-N/2}^{N/2-1} X_N(k)\mathfrak{z}^k$$

is its z-transform. For $\mathfrak{z} = \exp(j2\pi(n + jm)/N)$, we get the complex-lag signal $x(n + jm)$ (see Fig. 2) in a dual form.

C. Discrete Complex-Lag Distribution as a Corrected Wigner Distribution

According to (21), we have

$$\begin{aligned} c(n, m) &= x^{-j}(n + jm)x^j(n - jm) \\ &= e^{j \ln|x(n-jm)/x(n+jm)|}. \end{aligned} \tag{30}$$

For any complex number, we can write $z = |z|\exp\{j\arg\{z\}\}$, $z^j = \exp(j\ln|z|)\exp(-\arg\{z\})$. For the complex number, $c(n, m) = (x(n - jm)/x(n + jm))^j$ holds $|c(n, m)| = 1$ for the signals $x(n)$ with constant amplitude. If the signal's amplitude is time varying, we can omit the term corresponding to $\exp(-\arg\{z\})$ in (30) since we have introduced the complex-lag term in order to correct the phase function only.

In this way, the complex-lag distribution can be calculated as a corrected form of the WD (22)

$$CTD(n, k) = 2WD(n, k) *_k FT_m\{c(n, m)\}. \tag{31}$$

This relation may be used for an efficient realization of a reduced interference form of the complex-lag distribution.

IV. Reduced Interferences Realization for Multicomponent Signals

Next, we will present a procedure that can be generalized for application of the complex-lag distribution on the multicomponent signals.

Consider a multicomponent signal

$$x(n) = \sum_{p=1}^{P} x_p(n)$$

whose components are well separated in the frequency direction for a given time instant n. Then, the short-time Fourier transform (STFT) of $x(n)$ is of the form

$$\begin{aligned} |STFT_x(n, k)|^2 &= |FT_m\{w(m)x(n+m)\}|^2 \\ &= \sum_{p=1}^{P} |STFT_{x_p}(n, k)|^2. \end{aligned} \tag{32}$$

The algorithm for cross-terms reduced realization of the complex-lag distribution is presented in the sequel.

Algorithm:

1) Calculate the cross-terms reduced pseudo WD by using the method [26], [27], [29], [31], which will be referred to as the S-method

$$SM(n, k) = \sum_{l=-L}^{L} STFT_x(n, k+l)STFT_x^*(n, k-l). \tag{33}$$

The value of L is such that a few samples of the STFT around frequency k are taken for calculation. Details about L are given within the examples and the mentioned references.

2) Calculate the correction term for the $CTD(n, k)$, based on (30).

The cross-terms reduced version of $c(n, m)$ is calculated in the following way.

a) For a given time instant n, find the position $k_1(n)$ of the transform $STFT_x(n, k)$ maximum

$$k_1(n) = \arg\left\{\max_k STFT_x(n, k)\right\}. \tag{34}$$

b) Consider the region $[k_1(n) - M_k, k_1(n) + M_k]$ around $k_1(n)$ as a signal component, and calculate (35), as shown at the bottom of the page. The maximal calculated value in the sums is related to the expected component width. It is of $\exp(2\pi mM_k/N)$ order. The cross-terms will be

$$\begin{aligned} c_1(n,m) &= w(m)e^{j\ln|x_1(n-jm)/x_1(n+jm)|} \\ &= w(m)e^{j(4\pi/N)mk_1(n)}\exp\left(j\ln\left|\frac{\sum_{k=-M_k}^{M_k} STFT_x(n,k+k_1(n))e^{j(2\pi/N)(n-jm)k}}{\sum_{k=-M_k}^{M_k} STFT_x(n,k+k_1(n))e^{j(2\pi/N)(n+jm)k}}\right|\right) \end{aligned} \tag{35}$$

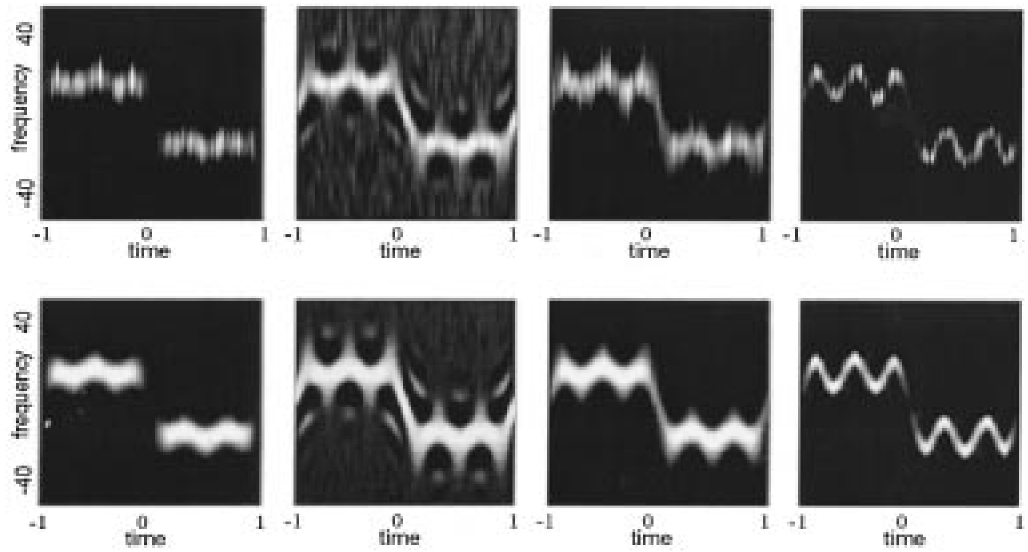

Fig. 3. TF representation of a signal with highly nonlinear instantaneous frequency. Spectrogram (first column), Wigner distribution (second column), S-method (third column), complex-lag distribution (fourth column). Upper row: signal with a relatively high noise such that $20\log(r/\sigma_\nu) = 10$ dB. Lower row: signal with a small noise such that $20\log(r/\sigma_\nu) = 30$ dB.

completely avoided if the distance between two auto-terms is at least $2M_k$ samples. The number of terms $2M_k + 1$ can also be adjusted to each signal component [29]. Then, the cross-term removal will be guaranteed for all components whose STFT do not overlap. This form of realization will be discussed in the next section as well. It is interesting to note that for the lag window of the form $w(m) = \exp(-(m/M_o)^4)$, we have $w(jm) = w(-jm) = w(m)$.

c) Put $STFT_x(n, k) = 0$ for $[k_1(n) - M_k, k_1(n) + M_k]$, and repeat the procedure from a) to c) P times, where P is the expected number of signal components. If the number of components P is not known, then the following procedure can be used. After the correction signal is calculated for the first component and the region around the transform $STFT_x(n, k)$ maximum, which is used in calculation of the correction component $c_1(n, m)$, is excluded, the maximum $\max_k |STFT_x(n, k)|$ in the remaining part of the frequency points k is determined. If this maximum is above an assumed level for a signal component, then the second correction component $c_2(n, m)$ is calculated for the frequency region around $k_2(n)$, where the second maximum is found. The procedure should be continued in this way until the value of $\max_k |STFT_x(n, k)|$, within the remaining frequency domain, is below the assumed level for a signal component.

The correction signal is then formed as

$$CF(n, k) = FT_m\left\{\sum_{p=1}^{P} c_p(n, m)\right\}. \tag{36}$$

3) Convolve results from 1) and 2), according to (31), in

$$CTD(n, k) = \sum_{l=-L}^{L} SM(n, k+l)CF(n, k-l). \tag{37}$$

The cross-terms reduced complex-lag distribution is obtained for the given time instant n.

V. Influence of Noise on the IF Estimation

Consider the complex-lag distribution of a signal $s(n) = re^{j\phi(n)}$ corrupted by a Gaussian white noise $\nu(n)$ [1],

$$\begin{aligned} CTD(n, \omega) = &\sum_{m=-N/2}^{N/2-1} w(m)(s(n+m) + \nu(n+m)) \\ &\times (s^*(n-m) + \nu^*(n-m))e^{-j2\Theta(n,m)}e^{-j4\omega m} \end{aligned} \tag{38}$$

where $\Theta(n, m)$ is defined, according to (30), by

$$\Theta(n, m) = \tfrac{1}{2}\ln|x(n+jm)/x(n-jm)|.$$

In order to simplify the notation, assume that the signal is of unity amplitude, when, according to (14), we can write

$$\begin{aligned} \Theta(n, m) &= \ln|x(n+jm)| \\ &= \ln|s(n+jm) + \nu(n+jm)]|. \end{aligned} \tag{39}$$

For small noise, the following approximation could be used:

$$\Theta(n, m) = \ln|s(n+jm)| + \Delta\Theta_\nu(n, m) \tag{40}$$

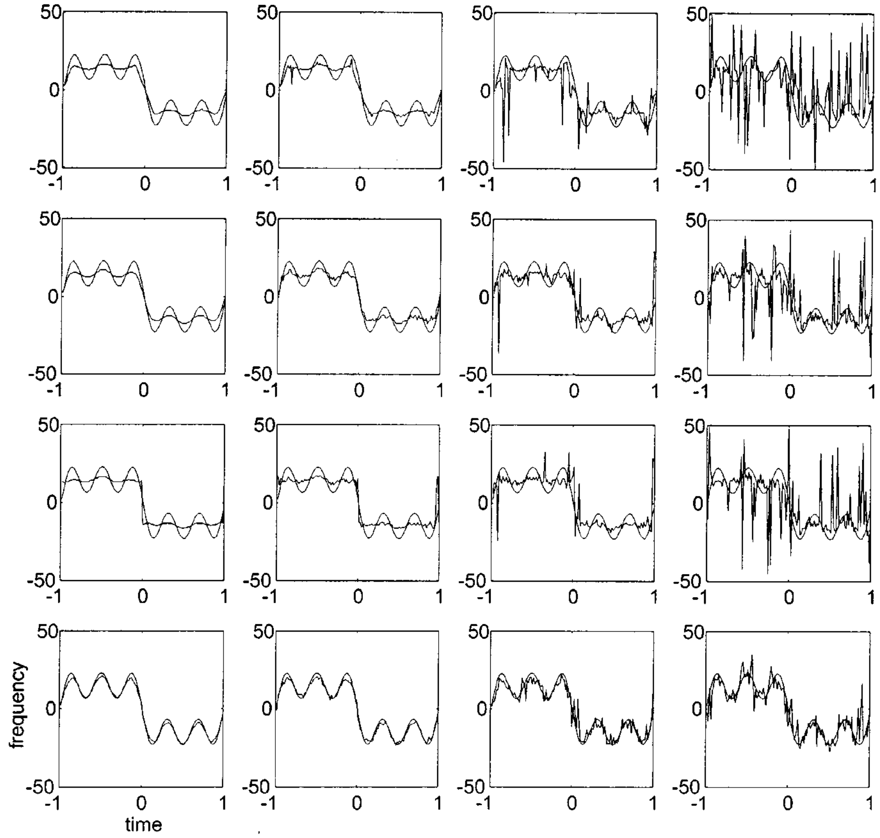

Fig. 4. IF estimation based on Wigner distribution (first row), S-method (second row), spectrogram (third row), complex lag distribution (fourth row). First column: $20\log(r/\sigma_\nu) = 30$ dB. Second column: $20\log(r/\sigma_\nu) = 10$ dB. Third column: $20\log(r/\sigma_n u) = 3$ dB. Fourth column: $20\log(r/\sigma_\nu) = 0$ dB.

where

$$\begin{aligned}\Delta\Theta_\nu(n, m) &= \text{Re}\left\{\frac{\nu(n+jm)}{s(n+jm)}\right\} \\ &= \text{Re}\left\{\frac{\sum_{k=k_1(n)-M_k}^{k_1(n)+M_k}\left[STFT_\nu(n,k)e^{j2\pi(n+jm)k/N}\right]}{\sum_{k=k_1(n)-M_k}^{k_1(n)+M_k}\left[STFT_s(n,k)e^{j2\pi(n+jm)k/N}\right]}\right\}\end{aligned} \tag{41}$$

is a phase deviation of the correction term caused by a small noise.

The IF is estimated as

$$\hat{\omega}_i(n) = \arg\left\{\max_n CTD(n, \omega)\right\}. \tag{42}$$

The estimation variance is (see the Appendix)

$$\begin{aligned}\text{var}\{\hat{\omega}_i(n) - \phi'(t)\} &= \text{var}\{\Delta\omega\} \\ &\cong \frac{\sigma_\nu^2}{8r^2M_2^2}\left(\sum_{m=-N/2}^{N/2-1} m^2w^2(m)\right.\end{aligned}$$

TABLE II
MEAN SQUARED ERROR OF THE INSTANTANEOUS FREQUENCY ESTIMATION BY USING VARIOUS DISTRIBUTIONS AND SIGNAL TO NOISE RATIO

Mean squared error	$\sigma_\nu = 0.031$	$\sigma_\nu = 0.31$	$\sigma_\nu = 0.707$	$\sigma_\nu = 1$
Wigner distribtution	26.69	28.96	120.40	440.16
S-method	21.11	23.18	76.95	267.73
Spectrogram	32.18	40.01	81.62	325.75
Complex-lag distribution	3.50	4.59	24.06	54.28

$$\left.+\frac{1}{N}\sum_{k=-M_k}^{M_k}\left(\sum_{m=-N/2}^{N/2-1} mw(m)e^{-(2\pi/N)km}\right)^2\right). \tag{43}$$

In order to keep the variance low, we have to keep M_k as low as possible. Thus, we can do either of the following.

1) Use the correction interval $[k_1(n) - M_k, k_1(n) + M_k]$ of a constant width such that the signal component is within this interval.
2) Assume, for each time instant n, a level for the correction interval $M_k = M_k(n, k)$ determination as a fraction of the maximal STFT value at that instant $R_n = \lambda \max\{|STFT_x(n, k)|\}$, where $0 < \lambda < 1$ is a constant [29]. Summation in (35) is done until $|STFT_x(n, k_1(n) + k)| \geq R_n$. For example, $\lambda = 0.1$ means that, in the correction calculation (35), the algorithm will take all STFT values around the considered point (n, k), where $|STFT_x(n, k)|$ is greater than 10% of the maximal value for that component. Two extreme

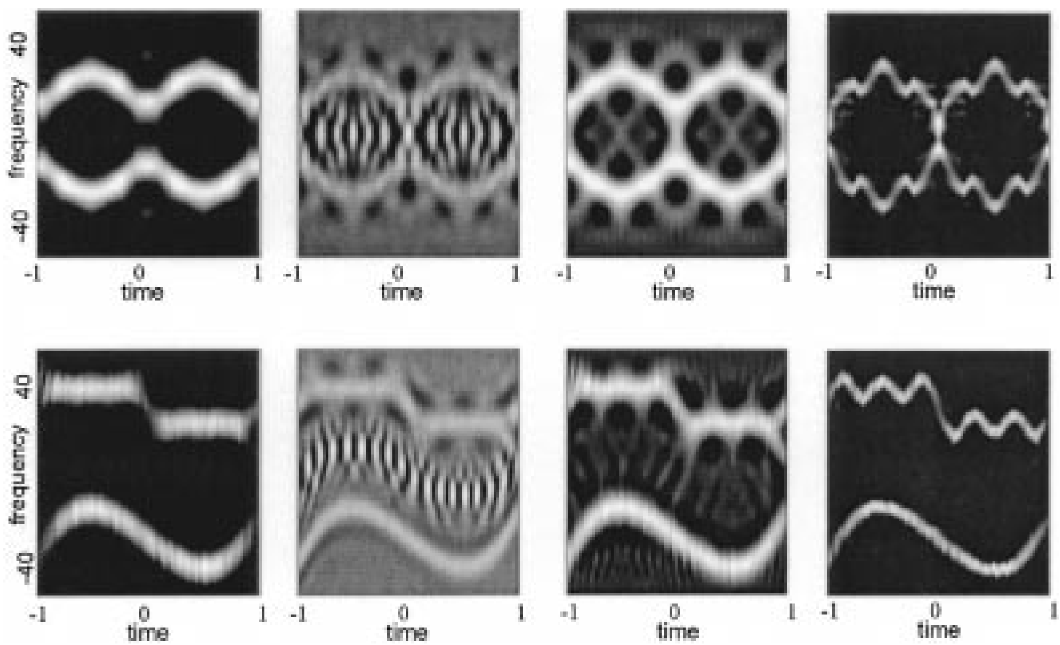

Fig. 5. TF representation of a multicomponent signal with highly nonlinear instantaneous frequency: Spectrogram (first column), Wigner distribution (second column), S-method (third column), complex-lag distribution (fourth column). Upper row: noisy signal with $20\log(r/\sigma_\nu) = 30$ dB. Lower row: noisy signal with $20\log(r/\sigma_n u) = 20$ dB.

cases are a) For $\lambda = 1$, the correction would not be done, thus producing calculations corresponding to the spectrogram; and b) for $\lambda = 0$, all points would be included, corresponding to the WD. For FM signals, according to the stationary principle [7], the auto-term form is defined by the lag window form and the IF derivative [33]. Thus, we can conclude that the auto-term transition along the frequency axis is fast, meaning that the method will not be too sensitive on the variations of parameter λ.

In both cases, the variance $\text{var}\{\hat{\omega}_i(n) - \phi'(t)_{|t=n\Delta t}\}$ could be slightly increased with respect to the WD. Having in mind that the bias is decreased for several orders of magnitude, we can expect a significant IF estimation improvement.

VI. Examples

Example 1: Consider a noisy monocomponent signal

$$x(t) = \exp\left[j(6\cos(\pi t) + \tfrac{2}{3}\cos(3\pi t) + \tfrac{2}{3}\cos(5\pi t))\right] + \nu(t) \tag{44}$$

within the interval $t \in [-1, 1]$, with $\Delta t = 2/N$, and $N = 128$. The STFT calculation is done according to

$$STFT_x(n, k) = FT_m\{x(n\Delta t + m\Delta\tau)w(m)\}. \tag{45}$$

The sampling rate for the lag coordinate $\Delta\tau = 1/16$ allows maximal signal frequency $\omega_m = \pi/\Delta\tau = 16\pi$. Note that the maximal IF in the signal is $\max\{\omega_i(t)\} = 11.3\pi$. The window of the form $w(m) = \exp(-(m\Delta\tau/T)^4)$ is used, with $2T = 1/2$. For the correction term $c(n, m)$, (35), which is the signal is oversampled by a factor of 2

$$STFT_{xc}(n, k) = FT_m\{x(n\Delta t + m\Delta\tau/2)w_c(m)\} \tag{46}$$

with $w_c(m) = \exp(-(m\Delta\tau/2/T)^4)$. The correction interval, which is defined by M_k, and the number of terms L in the S-method calculation, was $L = M_k = 16$. Note that, due to the fact that the lag windows are narrower than the whole considered lag interval, this value for L and M_k means only a few significant samples in the correction calculations. The same sampling interval $\Delta\tau/2$ should be used for the WD calculation, according to

$$WD(n, k) = FT_m\left\{x\left(n\Delta t + m\,\frac{\Delta\tau}{2}\right)x^*\left(n\Delta t - m\,\frac{\Delta\tau}{2}\right)w_c(m)\right\}. \tag{47}$$

The WD, the S-method, the spectrogram, and the complex-lag distribution are shown in Fig. 3. Based on the distributions from Fig. 3, the IF is estimated for various values of the noise standard deviation: $\sigma_\nu = 0.031$, $\sigma_\nu = 0.31$, $\sigma_\nu = 0.707$, and $\sigma_\nu = 1$. The estimated IF is shown in Fig. 4. We can see that although the lag window is quite narrow, the bias in the WD and the spectrogram is significant and dominates in the estimation error. Variance in the complex-lag distribution is slightly higher, whereas the bias is significantly lower, thus improving the overall estimation. When the noise is increased, the number of instants where the IF estimator completely misses the IF increases [13] in the WD and the spectrogram. Therefore, although the variance of the estimation should not be large, the misses degrade the performance of these distributions.

Mean square errors calculated in 128 realizations are presented in Table II. The main difference between the WD and the S-method comes from a small cross-term that starts to appear in sharp IF transition region. There, the S-method behavior is between the behavior of the WD and the spectrogram. However, it is far from the complex-lag distribution performance in all considered cases.

Example 2: Multicomponent signal representation will be illustrated next. Consider the signal

$$x(t) = \exp(j7.5\pi(0.5t^4 - 0.8t^2) - j8.5\pi t) + \exp(j(3\cos(\pi t) + \cos(3\pi t)/2 + \cos(5\pi t)/2 + 8.5\pi t)) + \nu(t) \quad (48)$$

with all discretization parameters as in the first example, and $\sigma_\nu = 0.1$. According to the procedure for a multicomponent signal realization, we get the representations shown in Fig. 5.

The case of real-valued signal

$$x(t) = \cos\left[9\cos(\pi t) + \tfrac{2}{3}\cos(3\pi t) + \tfrac{5}{7}\cos(5\pi t)\right] + \nu(t) \quad (49)$$

whose components intersect, is shown in Fig. 5, as well. Cross-terms between positive and negative frequencies are removed in the same way as other cross-terms.

VII. Conclusion

The complex argument distributions are proposed and analyzed. It has been shown that the inner artifacts in the representation of signals with fast varying frequency or group delay could be significantly reduced. A procedure for the reduced interferences (noise and cross-terms) realization of the complex-lag distribution is presented. The complex-lag distribution as an IF estimator is analyzed. The procedure for realization, and its application in the IF estimation, is demonstrated on numerical examples. Further research could be directed toward a wider application of the complex-lag argument in other TF, and not only TF, problems.

Appendix
IF Estimation Variance and Bias

The IF of $x(t) = r\exp(j\phi(t)) + \nu(t)$ at $t = nT$ is estimated by using the complex-lag distribution (38) and (42).

Assuming that all disturbances are small, then the linearization of $\partial CTD(n, \omega)/\partial\omega$ around the position where the distribution maximum is detected (denoted by subscript 0) gives [34]

$$\frac{\partial CTD(n, \omega)}{\partial\omega}_{|0} + \frac{\partial^2 CTD(n, \omega)}{\partial\omega^2}_{|0}\Delta\omega + \frac{\partial CTD(n, \omega)}{\partial\omega}_{|0}\delta_{\Delta Q} + \frac{\partial CTD(n, \omega)}{\partial\omega}_{|0}\delta_\nu = 0. \quad (50)$$

Here, we assume that all sources of errors are small; therefore, a linear model with respect to each one of them separately can be used. The possible sources of errors are

- the window form [first and second term in (50)];
- the bias caused by the signal form (third term);
- the noise (the last term).

1) The distribution first derivative of (38) at the stationary point when there is no disturbance is

$$\frac{\partial CTD(n, \omega)}{\partial\omega}_{|0} = -\sum_{m=-N/2}^{N/2-1} 4jr^2 m w(m).$$

It is equal to zero for a symmetric window $w(m)$, where $\hat{\omega}_i(n)$ is equal to the correct IF.

2) The second derivative of the distribution when there is no signal or noise disturbance is

$$\frac{\partial^2 CTD(n, \omega)}{\partial\omega^2}_{|0}\Delta\omega = -\sum_{m=-N/2}^{N/2-1} 16|r|^2 m^2 w(m)\Delta\omega = -16r^2 M_2\Delta\omega \quad (51)$$

where

$$M_k = \sum_{m=-N/2}^{N/2-1} m^k w(m), \qquad k = 2, 3, 4, \ldots \quad (52)$$

are the moments of the lag window.

3) The distribution derivative around the stationary point when there is only a small disturbance caused by the signal spread factor $Q(n, m)$ is of the form

$$\frac{\partial CTD(n, \omega)}{\partial\omega}_{|0}\delta_Q = \sum_{m=-N/2}^{N/2-1} 4mQ(n, m)w(m)r^2 = \phi^{(5)}(n + n_1)\frac{16M_6}{5!}r^2 \quad (53)$$

since $e^{jQ(n, m)} \cong 1 + jQ(n, m)$. This factor causes the IF estimation bias. The notation $\phi^{(5)}(n)$ means $d^5\phi(t)/dt^5$ at $t = nT$.

4) The last term is the distribution derivative variation around the stationary point caused by a small noise $\nu(n)$. Then, the component due to noise in $\Theta(n, m)$, which is denoted by $e^{-j2\Delta\Theta_\nu(n, m)} \cong 1 - j2\Delta\Theta_\nu(n, m)$, also exists. This term has the form

$$\frac{\partial CTD(n, \omega)}{\partial\omega}_{|0}\delta_\nu \cong \sum_{m=-N/2}^{N/2-1} j4mw(m) \times [s(n+m)s^*(n-m)j2\Delta\Theta_\nu(n, m) - s(n+m)\nu^*(n-m) - \nu(n+m)s^*(n-m)] \times e^{-j2\phi'(n)m}. \quad (54)$$

It causes random variations of the estimated IF. The higher order noise terms are neglected.

According to (50), (51), and (53), *the bias* is

$$bias(\Delta\omega) = \frac{1}{M_2}\left(\phi^{(5)}(n)\frac{M_6}{5!} + \phi^{(9)}(n)\frac{M_{10}}{9!} + \cdots\right). \quad (55)$$

For an ideally concentrated distribution (producing the unbiased IF estimator), the factor $Q(n, m)$ should be equal to zero (see Table I).

Variance of the IF estimate, according to (50), is

$$\begin{aligned}\text{var}(\Delta\omega) &= \frac{E\left\{\left|\frac{\partial CTD(n,\,\omega)}{\partial\omega}_{|0}\delta_\nu\right|^2\right\}}{\left(\frac{\partial^2 CTD(n,\,\omega)}{\partial\omega^2}_{|0}\right)^2} \\ &= \frac{1}{16r^4M_2^2}\sum_{m_1=-N/2}^{N/2-1}\sum_{m_2=-N/2}^{N/2-1} \\ &\quad\times m_1 m_2 w(m_1)w(m_2)\times e^{-j2\phi'(n)(m_1-m_2)} \\ &\quad\times [E\{4\Delta\Theta_\nu(n,\,m_1)\Delta\Theta_\nu^*(n,\,m_2) \\ &\qquad\times s(n+m_1)s^*(n-m_1)s^*(n+m_2) \\ &\qquad\times s(n-m_2)+[s(n+m_1)\nu^*(n-m_1) \\ &\qquad+\nu(n+m_1)s^*(n-m_1)] \\ &\qquad\times [s^*(n+m_2)\nu(n-m_2)+\nu^*(n+m_2) \\ &\qquad\quad\times s(n-m_2)]\}]. \end{aligned} \tag{56}$$

According to (41), and $\text{Re}\{z\}\text{Re}\{w^*\} = 1/2\text{Re}\{zw^*\} + 1/2\text{Re}\{zw\}$, we have

$$\begin{aligned} V &= E\{\Delta\Theta_\nu(n,\,m_1)\Delta\Theta_\nu^*(n,\,m_2)\} \\ &= \frac{1}{2}\,\text{Re}\left\{E\left\{\frac{\nu(n+jm_1)}{s(n+jm_1)}\,\frac{\nu^*(n+jm_2)}{s^*(n+jm_2)}\right\}\right\} \\ &\quad+\frac{1}{2}\,\text{Re}\left\{E\left\{\frac{\nu(n+jm_1)}{s(n+jm_1)}\,\frac{\nu(n+jm_2)}{s(n+jm_2)}\right\}\right\}. \end{aligned}$$

Since the model assumes independent error analysis, we may take that the signal itself is not disturbed by the finite summation in (41), i.e., that it takes the form $s(n) = r\exp(j2\pi nk_1(n)/N)$. The analysis will be simplified by assuming a rectangular window in the initial STFT that is used for the calculation of signal with complex argument. Then, $E\{STFT_\nu(n,\,k_1)STFT_\nu^*(n,\,k_2)\} = N\sigma_\nu^2\,\delta(k_1-k_2)$, and $E\{STFT_\nu(n,\,k_1)STFT_\nu(n,\,k_2)\} = 0$. With this assumption, we get

$$V = \frac{\sigma_\nu^2}{2Nr^2}\sum_{k=-M_k}^{M_k} e^{-(2\pi/N)k(m_1+m_2)}.$$

Therefore, neglecting the error cross-covariance terms, the variance is obtained in form of (43).

For a rectangular window $w(m)$ of the width N_w in (38), the first term is of $N_w^3/12$ order, whereas the second term is of $(N_w/2)^2\exp(2\pi N_w M_k/N)/(2\pi)$ order; thus, we get

$$\text{var}\{\Delta\omega\} \sim \frac{\sigma_\nu^2 N_w^3}{24r^2M_2^2}\left(\frac{1}{4}+\frac{3}{8\pi N_w}\,e^{2\pi N_w M_k/N}\right). \tag{57}$$

The variance in complex-lag distribution is of the same order as the variance in the WD, as far as $3\exp(2\pi N_w M_k/N)/(8\pi N_w) \sim 1$, since the frequency in (38) is scaled by a factor of 2 [19]. This is the case, for example, for $N = 128$, $2M_k = 16$, and $N_w = 16$. By using signal-dependent M_k, as it is described in the last paragraphs of Section V, then its value could be quite small, resulting in a small variance and robust IF estimation. However, a direct calculation of the complex argument distributions, corresponding to a full range of $M_k = N/2$, will result in a representation that is very sensitive to additive noise.

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