General Form of Time-Frequency Distribution with Complex-lag Argument

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Abstract— **A general form of the time-frequency distribution with complex-lag argument is proposed. It is based on the Generalized complex-lag distribution, modified to provide an efficient instantaneous frequency estimation in the case of multicomponent signals. The form of proposed distribution is suitable for numerical realization and it provides an arbitrarily high distribution concentration. The theory is illustrated by an example.**

I. Introduction

The distribution concentration is very important for time-frequency signal analysis. The commonly used Wigner distribution provides an ideal concentration along the linear instantaneous frequency (IF). To improve distribution concentration for signals with non-linear IF, the L-Wigner distribution and polynomial Wigner distribution have been used [1]-[3]. By introducing time-frequency distribution with complex-lag argument [4], further improvement of distribution concentration is achieved. This distribution has been studied in [5]-[7]. Advantages of the complex time-frequency distribution are especially emphasized in the case of a fast IF variations, within several signal samples. The generalized complex-lag distribution (GCD) has been recently introduced and analyzed for the case of monocomponent signals [8]. It has been shown that the GCD provides an arbitrarily high concentration by increasing the distribution order. In this letter, the time-frequency GCD is modified to provide a general form of cross-terms free representation for multicomponent signals with fast varying IF. Theoretical considerations are illustrated and proven by the example.

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II. Theory

Time-frequency representation for a signal in the form $x{=}Ae^{j\phi(t)}$ can be, generally, written as:

$$TFR(t,\omega) = = 2\pi A^2\delta(\omega-\phi'(t))_{*\omega}W(\omega)_{*\omega}FT\{e^{jQ(t,\tau)}\}, \quad (1)$$

where $W(\omega)$ is the Fourier transform (FT) of a lag-window, and $Q(t,\tau)$ is the factor causing distribution spread around the IF. An ideal time-frequency representation would be obtained for $Q(t,\tau){=}0$ (the artifacts do not exist).

The time-frequency GCD has been defined as [8]:

$$GCD_N(t,\omega) = = \int_{-\infty}^{\infty}\prod_{p=0}^{N-1} x^{w^*_{N,p}}(t+\frac{w_{N,p}}{N}\tau)e^{-j\omega\tau}d\tau, \quad (2)$$

where $w_{N,p}{=}\mathrm{e}^{2j\pi p/N}$ are the roots on the unit circle (* denotes complex conjugate), while N is an even number that represents the distribution order. Note that the roots on the unit circle appear in pairs: $w_{N,p+N/2}{=}\text{-}w_{N,p}$.

The spread factors for some time-frequency distributions are given in Table I.

To make it suitable for multicomponent signal analysis, we introduce a modification of (2):

$$GCD_N(t,\omega) = = \int_{-\infty}^{\infty} x(t+\frac{\tau}{N})x^*(t-\frac{\tau}{N})c(t,\tau)e^{-j\omega\tau}d\tau, \quad (3)$$

where,

$$c(t,\tau) =$$

TABLE I
Spread factors in some time-frequency distributions

Distribution	**Spread factor**
[1ex] Wigner distribution, (GCD_N, N=2)	$Q(t,\tau)=\phi^{(3)}(t)\frac{\tau^3}{2^2 3!}+\phi^{(5)}(t)\frac{\tau^5}{2^4 5!}+...$
Polynomial Wigner distribution (IV ord.)	$Q(t,\tau)=-0.327\phi^{(5)}(t)\frac{\tau^5}{5!}-0.386\phi^{(7)}(t)\frac{\tau^7}{7!}+...$
[1ex] GCD_N, N=6	$Q(t,\tau)=\phi^{(7)}(t)\frac{\tau^7}{6^6 7!}+\phi^{(13)}(t)\frac{\tau^{13}}{6^{12}13!}+...$
GCD_N, N=s	$Q(t,\tau)=\phi^{s+1}(t)\frac{\tau^{s+1}}{s^s(s+1)!}+\phi^{2s+1}(t)\frac{\tau^{2s+1}}{s^{2s}(2s+1)!}+...$

$$=\prod_{p=1}^{N/2-1} x^{w^*_{N,p}}(t+\frac{w_{N,p}}{N}\tau)x^{-w^*_{N,p}}(t-\frac{w_{N,p}}{N}\tau). \quad (4)$$

The quantity c(t,τ) will be referred as the concentration function. Note that this function arbitrarily improves the concentration of the Wigner distribution; namely, by increasing distribution order N, the influence of inner interference terms ($Q(t,\tau)$) will be significantly reduced (Table I). Also, this form of the time-frequency GCD satisfies the marginal properties. Observe that the GCD_2(for N=2) represents the Wigner distribution (with the auto-correlation function $R_t(2\tau/N)=x(t+\tau/N)\ x^*(t-\tau/N)$).

To modify the concentration function c(t,τ), the term $w_{N,p}$, will be written in the form: $w_{N,p}=w_{r_p}+jw_{i_p}$, where w_{r_p} and w_{i_p} are the real and the imaginary part of $w_{N,p}$, respectively. Thus, $c(t,\tau)$ for $\forall p$: p=1,...,N/2-1, can be written as:

$$c_p(t,\tau)=R_t^{w_{r_p}}(\frac{2(w_{r_p}+jw_{i_p})}{N}\tau)$$
$$\times R_t^{-jw_{i_p}}(\frac{2(w_{r_p}+jw_{i_p})}{N}\tau)=$$
$$=c_{r_p}(t,\tau)c_{i_p}(t,\tau), \quad (5)$$

where $R_t(\cdot)$ corresponds to the autocorrelation function. Considering the signal of the form x=$Ae^{j\phi(t)}$, and applying the Taylor series expansion of the phase function, we obtain:

$$c_{r_p}(t,\tau)=$$
$$=e^{jw_{r_p}(\phi(t+(w_{r_p}+jw_{i_p})\frac{\tau}{N})-\phi(t-(w_{r_p}+jw_{i_p})\frac{\tau}{N}))}$$
$$=e^{2jw_{r_p}(\phi'(t)w_{r_p}\frac{\tau}{N}+\phi^{(3)}(t)(w_{r_p}^3-3w_{r_p}w_{i_p}^2)\frac{\tau^3}{3!N}+...)}$$
$$\times e^{-2w_{r_p}(\phi'(t)w_{i_p}\frac{\tau}{N}+\phi^{(3)}(t)(3w_{r_p}^2w_{i_p}-w_{i_p}^3)\frac{\tau^3}{3!N}+...)} \quad (6)$$

Thus, $c_{r_p}(t,\tau)$ contains two terms. The first one brings the information about the IF, while the second represents the amplitude modulation. The value of this term can be large and it might affect the precision of the IF estimation. To avoid the influence of the amplitude term, the following modification is introduced:

$$c'_{r_p}(t,\tau)=e^{jw_{r_p}angle(R_t(\frac{2(w_{r_p}+jw_{i_p})}{N}\tau))}. \quad (7)$$

By analogy with $c_{r_p}(t,\tau)$ (when the exponent $-jw_{i_p}$ is used in (6) instead of w_{r_p}), the modification of $c_{i_p}(t,\tau)$ can be written as:

$$c'_{i_p}(t,\tau)=e^{-jw_{i_p}\log\left|R_t(\frac{2(w_{r_p}+jw_{i_p})}{N}\tau\right|}. \quad (8)$$

Replacing $c_{r_p}(t,\tau)$ and $c_{i_p}(t,\tau)$ in (5) with (7) and (8), respectively, and having in mind (4) and (3), the GCD can be modified as:

$$MGCD_N(t,\omega)=$$
$$=FT_\tau\{R_t(\frac{2}{N}\tau)\prod_{p=1}^{N/2-1}c'_{r_p}(t,\tau)c'_{i_p}(t,\tau)\}$$
$$=\frac{N}{2}WD(t,\frac{N}{2}\omega)_{*\omega}FT_t\left\{c(t,\tau)\right\}, \quad (9)$$

where $c(t,\tau)=\prod_{p=1}^{N/2-1}c'_{r_p}(t,\tau)c'_{i_p}(t,\tau)$ (FT_τ is the Fourier transform), while WD is the Wigner distribution.

III. A general form of discrete MGCD for multicomponent signals

The distribution defined by (9) will be further modified for the case of multicomponent signals $x(n)=\sum_{q=1}^{Q}A_q e^{j\phi_q(n)}$, providing a general form of the cross-terms free time-frequency distribution with complex-lag argument.

Similarly as in [5], the analytical extension of $x(n)$ for multicomponent signals can be written as eq. 10.

where n and k are discrete time and frequency variables, respectively, while $k_q(n) = \arg\left\{\max_k STFT(n,k)\right\}$ is the position of the qth signal component maximum in the short time Fourier transform ($STFT$), for a given instant n. It is assumed that the qth signal component is within the region [$k_q(n)$-W_q, $k_q(n)$+W_q]. Thus, for a given instant n, $c'_{r_p}(n,m)_q$ and $c'_{i_p}(n,m)_q$ of the qth signal component, are calculated according to eq. 11 and eq. 12.

In the next iteration $STFT(n,k)$ should be set to zero within the region [$k_q(n)$-W_q, $k_q(n)$+W_q]. The procedure is repeated Q times, where Q is the expected number of signal's components. It is important to note that this approach removes the cross-terms if the distance between signal's components is d=2W_q. Finally, the Fourier transforms of $c'_{r_p}(n,m)_q$and $c'_{i_p}(n,m)_q$ are:

$$C_r(n,k)=FT_m\left\{\prod_{p=1}^{N/2-1}\sum_{q=1}^{Q}c'_{r_p}(n,m)_q\right\}$$

$$C_i(n,k)=FT_m\left\{\prod_{p=1}^{N/2-1}\sum_{q=1}^{Q}c'_{i_p}(n,m)_q\right\}. \tag{13}$$

By introducing the S-method instead of the Wigner distribution in (9), and performing the convolution around l=0, the modified general form of cross terms free complex time-frequency distribution is defined as:

$$MGCD_N(n,k)=$$

$$=\sum_{l=-L}^{L}P(l)SM(n,k+l)C(n,k-l), \tag{14}$$

where the $SM(n,k)$ is, [1]: $SM(n,k)=\sum_{l=-L}^{L}P(l)STFT(n,k+l)STFT^*(n,k-l)$, and $C(n,k)=\sum_{l=-L}^{L}P(l)C_r(n,k+l)C_i(n,k-l)$. Note that the window $P(l)$ will remove all cross-terms if the size of window is less than the minimal distance between auto-terms.

Example: Consider a multicomponent signal in the form:

$$x(t)=e^{j\cdot(3\cdot\cos(\pi t)+2/3\cdot\cos(5\pi t)-6.5\pi t)}+$$

$$+e^{j\cdot(4\cdot\cos(0.5\pi t)+3/2\cdot\cos(0.5\pi t)+1/2\cdot\cos(5\pi t)+8.5\pi t)}, \tag{15}$$

where t∈[-1,1] and $\nu(t)$ is complex Gaussian white noise with $\sigma_\nu=0.175$ (the SNR is 15 dB, and is lower than the SNR considered in [15] for multicomponent signals). The signal is sampled at $A/t=1/64$, and L=5 is used. For the calculation of the concentration functions $c'_{r_p}(n,m)$ and $c'_{i_p}(n,m)$ the signal is oversampled by a factor $N/2$.

The results obtained for MGCD with N=2, 4, and 6 are given in Fig 1. Note that for

$$x_q(n\pm(w_{r_p}+jw_{i_p})\frac{m}{N})=\sum_{-W_q}^{W_q}STFT(n,k+k_q(n))e^{j(k+k_q(n))(n\pm(w_{r_p}+jw_{i_p})\frac{m}{N})}, \tag{10}$$

$$c'_{r_p}(n,m)_q=e^{2j\frac{m}{N}k_q(n)(w_{r_p})^2}\exp\left(jw_{r_p}angle(\frac{\sum_{-W_q}^{W_q}STFT(n,k+k_q(n))e^{j(n+(w_{r_p}+jw_{i_p})\frac{m}{N})k}}{\sum_{-W_q}^{W_q}STFT(n,k+k_q(n))e^{j(n-(w_{r_p}+jw_{i_p})\frac{m}{N})k}})\right) \tag{11}$$

$$c'_{i_p}(n,m)_q=e^{2j\frac{m}{N}k_q(n)(w_{i_p})^2}\exp\left(jw_{i_p}\log\left|\frac{\sum_{-Wq}^{W_q}STFT(n,k+k_q(n))e^{j(n-(w_{r_p}+jw_{i_p})\frac{m}{N})k}}{\sum_{-Wq}^{W_q}STFT(n,k+k_q(n))e^{j(n+(w_{r_p}+jw_{i_p})\frac{m}{N})k}}\right|\right). \tag{12}$$

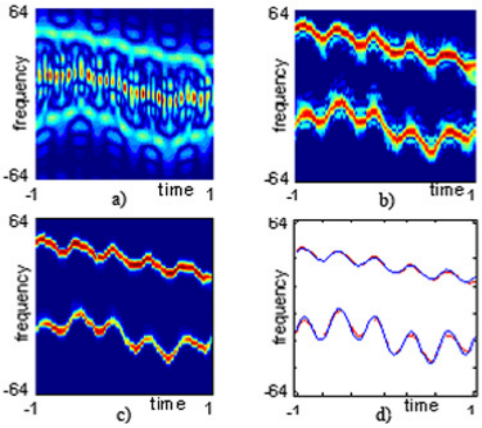

Fig. 1. a) MGCD, N=2 (Wigner distribution), b) MGCD, N=4, c) MGCD, N=6, d) exact IF (blue line) and estimated IF (red line) from MGCD with N=6

the considered signal, the MGCD with N=6 (Fig 1.c) produces significantly higher auto-terms concentration than the MGCD with N=4, while the Wigner distribution (Fig 1.a) is useless for the IF estimation. The exact IF and IF estimated from the MGCD with N=6 are given in Fig 1.d.

IV. Conclusion

The time-frequency generalized complex-lag distribution is modified. The introduced modifications lead to the general form of cross-terms free highly concentrated distribution that can be successfully used for estimation of the fast varying IF of multicomponent signals.

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