Two Approaches to Adaptation of Sample Myriad to Characteristics of SαS Distribution Data

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Abstract— **This paper deals with theoretical and numerical simulation based analysis of sample myriad properties for a family of symmetric α-stable (SαS) distributions often used for modeling noise in natural environments. The theoretically optimal values of a sample myriad tunable parameter k in the sense of minimal asymptotic variance are obtained. Two practical approaches are proposed to adapt the parameter k to SαS distribution characteristics for data samples of a limited size. Statistical properties of the developed approaches are studied for cases of a priori unknown parameters of SαS distribution. A practical application where the proposed approaches can be useful is considered.**

I. Introduction

For many decades, noise and other adverse phenomena in radars, acoustics, communications, and other fields have been described using Gaussian distribution [1]. However, recent studies have clearly demonstrated that processes with heavier tail probability density functions (PDFs) produce more adequate description of noise characteristics [2-6]. The family of SαS distributions is one type of PDFs widely used to model non-Gaussian environments [2-4, 6].

A SαS distribution with zero location parameter is characterized by two parameters, namely, parameter α ($0 < \alpha \leq 2$) that describes the heaviness of the PDF tail and parameter γ that describes a distribution scale. A PDF scale is determined by both aforementioned parameters as $\gamma^{1/\alpha}$ [7, 8]. The family of SαS distributions includes a Gaussian PDF as a marginal case for α=2 with variance $\sigma_G^2 = 2\gamma$. Another known distribution that belongs to the SαS family is a Cauchy PDF observed for α=1, with a scale given by the parameter γ. For other α, there are no explicit expressions to describe SαS distribution. Such PDFs are commonly expressed by characteristic functions as $\varphi(\alpha, \gamma; \omega) = -\gamma |\omega|^{\alpha}$ where ω is an argument in the characteristic function domain.

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For signals embedded in non-Gaussian noise, different robust estimation methods [9] are widely used [2, 10]. Standard median and α-trimmed mean filters [10] are, probably, the most known examples. Recently, various adaptive nonlinear filters and robust DFT (RDFT) based methods have been designed to deal with non-Gaussian and impulsive noise [11-14] where robust estimators are used.

A desirable feature of a robust estimator for signal and image processing applications is its ability to adapt to noise statistics (random data) since statistical characteristics are often not known in advance and can vary [15, 16]. In this case, the sample myriad estimator introduced by G. Arce and J. Gonzalez [3, 17] is quite attractive due to availability of freely tunable parameter k. In general, this estimator has several useful properties. The sample myriad estimator belongs to the class of M-estimators. It is maximum likelihood estimator for Cauchy distribution under the condition k=γ. Recently, it has been also demonstrated [18] that the sample myriad estimator with optimally set k is the quasi-optimal estimator for the entire family of SαS distributions. Main properties of the sample myriad estimator can vary in wide range by selecting different k values. For relatively small value of k parameter, myriad is able to perform as an efficient mode finder not only for symmetric but also for asymmetric distributions [19, 20]. On the contrary, for relatively large k the sample myriad estimator approaches to the sample mean one.

However, the "relatively small" and "relatively large" values are not known exactly. There is only an assumption that k should depend upon the distribution scale. To select k correctly, we should, at least, have a priori information about underlying noise PDF [3, 7, 8, 18, 21, 22]. However, in many practical situations such information is not available. Then the important task is to design practical algorithms for myriad adaptation to noise characteristics.

Firstly the optimal α-k characteristic that establishes connection between optimal value k_{opt} (that provides minimal variance) for different α and fixed γ, was given in [3], later it was presented in journal papers [7, 22]. It was proven that for $\alpha \to 0$ the value k_{opt} should tend to 0; for α =1, i.e., for Cauchy distribution, k_{opt} has to be equal to γ, and, finally, for Gaussian PDF (α =2) the optimal k should tend to infinity. Meanwhile, the approximation of α-k characteristic as $k_{opt\,A}(\alpha,\, \gamma) = \sqrt{\alpha/(2-\alpha)} \cdot \gamma^{1/\alpha}$ put forward in [3] occurred to be incorrect. This has been independently shown recently in papers [18] and [8, 21]. Even for γ=1 the approximation $k_{opt\,A}(\alpha,\, \gamma)$ is not accurate. For γ>1 and small α (<1) this approximation produces $k_{opt\,A}(\alpha,\, \gamma)$ which considerably differ from the correct dependence $k_{opt}(\alpha,\, \gamma)$. In particular, it has been analytically shown in [8] that for γ>1 and $\alpha \to 0$ the values of $k_{opt\,A}(\alpha,\, \gamma)$ tend to infinity. Thus, one of the tasks is to provide correct theoretical dependence $k_{opt\,A}(\alpha,\, \gamma)$ valid for any α and γ.

Note that the parameters α and γ are often a priori unknown in practice. There is a method for estimating them [23] but it is quite complicated. There are also other methods [4], [24-27] that have been mainly tested for rather large sample sizes. Thus, there is a need in simple and accurate algorithms for determining $\hat{k}_{opt}$ for a given data sample of a limited size that is suitable for signal and image processing applications (several tens or hundreds). Initial steps in this direction have been made in [8] where three approaches to adaptation of sample myriad tunable parameter to characteristics of SαS distribution have been briefly considered and tested. Therefore, the goal of this paper is to carry out a more detailed theoretical analysis of the sample myriad estimator and to verify applicability of the corresponding practical algorithms.

The paper is organized as follows. In Section metricconverterProductID2, a2, a more detailed description of the myriad estimator is given. Maximum likelihood estimator for location of SαS distribution with given α and γ is presented and basic accuracy characteristics of such estimates are considered. The optimal values of k_{opt} for $\gamma \neq 1$ are found. Based on them a more accurate approximating expression for optimal k derivation is presented. Section 3 describes the developed adaptive approaches to determination of k for data samples of limited size. Special attention is paid to the accuracy of α and γ parameter estimation. The accuracy of adaptive myriad estimators is studied in Section 4 by analyzing the variance of obtained location estimates for SαS distributed data. Comparison of accuracy of proposed and optimal estimators for different α and γ are carried out under the condition of using proper k. In Section 5, we consider the application of the proposed adaptive estimation to robust filtering of signals embedded in non-Gaussian noise. Finally, the conclusions and possible directions of future research are presented.

II. Determination of optimal k

A. *Theoretical background*

For providing theoretical background for applying myriad estimator, consider the following model of an observed process

$$y_n = \theta + \xi_n,\; n = \overline{1, N}, \qquad (1)$$

where ξ_n is the n-th sample of noise that obeys SαS PDF $f_\alpha(\gamma; x)$ with zero location and aforementioned parameters α and γ; θ denotes unknown parameter.

Ì-estimator of θ for an observation corrupted by noise ξ_n with PDF f_α is defined as

$$\hat{\theta}_{ML} = \arg\min_{\theta \in \Theta} \left[-\sum_{n=1}^{N} \log f_\alpha(y_n - \theta) \right]. \quad (2)$$

For Gaussian distribution of ξ_n (α =2) the solution of (2) leads to the sample mean estimator. For Cauchy PDF (α=1) the Ì-estimator is a log-Cauchy filter expressed as

$$\hat{\theta}_{ML} = \arg\min_{\theta\in\Theta} \sum_{n=1}^{N} \log\left\{\gamma^2 + (y_n - \theta)^2\right\}. \quad (3)$$

In [3], this filter has been generalized by using tunable parameter k instead of γ. Thus, the myriad estimator can be described as

$$\hat{\theta}_{Myr} = \arg\min_{\theta\in\Theta} \sum_{n=1}^{N} \log\left\{k^2 + (y_n - \theta)^2\right\}. \quad (4)$$

The task (4) can be also expressed using cost function

$$\sum_{n=1}^{N} \psi_{Myr}(y_n - \hat{\theta}_{Myr}) = 0, \quad (5)$$

where $\psi_{Myr}(x) = d\rho(x)/dx$, $\rho(x) = \log(k^2 + x^2)$ denotes the myriad cost function.

As it is known [9], the location estimates for distribution function $F = \int_{-\infty}^{\infty} f_\alpha(\gamma;x)dx$ obtained by the M-estimator are consistent and asymptotically normal. Thus, the asymptotical variance of the sample myriad $\sigma^2_{Myr}\left(\psi_{Myr};F\right)$ can be defined as

$$\sigma^2_{Myr}\left(\psi_{Myr};F\right) = \frac{\int \psi^2_{Myr} dF}{\left(\int \psi'_{Myr} dF\right)^2}, \quad (6)$$

$$\psi'_{Myr}(x) = d\psi_{Myr}(x)/dx.$$

Substituting into (6), we obtain

$$\sigma^2_{Myr}\left(k; f_\alpha(\gamma;x)\right) = \frac{\int \frac{x^2}{(k^2+x^2)^2} f_\alpha(\gamma;x)dx}{\left(\int \frac{k^2-x^2}{(k^2+x^2)^2} f_\alpha(\gamma;x)dx\right)^2}. \quad (7)$$

Variance of Ì-estimator for the PDF $f_\alpha(\gamma;x)$ is determined by the following expression [9]:

$$\sigma^2_{ML}\left|_{(\alpha,\gamma)}\right. = \left[\int \frac{f'_\alpha(\gamma;x)^2}{f_\alpha(\gamma;x)} dx\right]^{-1}. \quad (8)$$

For Gaussian and Cauchy distributions, the values σ^2_{Myr} and σ^2_{ML} can be derived analytically. In the former case, minimal asymptotic variance of the myriad estimator is reached if $k \to \infty$. Then

$$\lim_{k\to\infty} \sigma^2_{Myr}\left(k; f_2(\gamma;x)\right) = 2\gamma, \quad (9)$$

and asymptotic variance of Ì-estimator for α=2 is

$$\sigma^2_{ML}\left|_{(\alpha=2,\gamma)}\right. = \left[\int \frac{f'_2(\gamma;x)^2}{f_2(\gamma;x)} dx\right]^{-1} = 2\gamma, \quad (10)$$

where $f_2(\gamma;x) = \frac{1}{2\sqrt{\pi\gamma}} \exp\left(\frac{-x^2}{4\gamma}\right)$ is Gaussian PDF which takes into account $\sigma^2_G = 2\gamma$.

Now consider the case when α=1, i.e., the Cauchy distribution is described as

$$f_1(\gamma;x) = \frac{\gamma}{\pi(\gamma^2 + x^2)}. \quad (11)$$

Then, as it has already been mentioned, the myriad estimator with k=γ is the Ì-estimator and its asymptotic variance is

$$\sigma^2_{ML}\left|_{(\alpha=1,\gamma)}\right. = \sigma^2_{Myr}\left(\gamma; f_1(\gamma;x)\right) =$$

$$= \left[\int \frac{f'_1(\gamma;x)^2}{f_1(\gamma;x)} dx\right]^{-1} = 2\gamma^2. \quad (12)$$

For any other α value there are no explicit expressions of the integral

$$f_\alpha(\gamma;x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\alpha,\gamma;\omega) e^{-j\omega x} d\omega. \quad (13)$$

Derivation of σ^2_{Myr} and σ^2_{ML} for $\alpha \neq 1$ and $\alpha \neq 2$ for SαS PDF has been performed numerically by using MatLab. The obtained dependence $f_\alpha(\gamma;x)$ then has been approximated by a polynomial and then a derivative has been calculated. RMSE between approximating polynomial and $f_\alpha(\gamma;x)$ (13) was about 10^{-7}. Numerical integration in (7) and (8) has been done by the method of trapezoids [28].

B. Analysis of theoretically obtained values σ^2_{Myr} and σ^2_{ML}

The derived values of σ^2_{Myr} and σ^2_{ML} for γ=1 coincide with data in [18]. This confirms the conclusion that the myriad estimator with optimal k is quasi-optimal estimator for the entire family of SαS PDF.

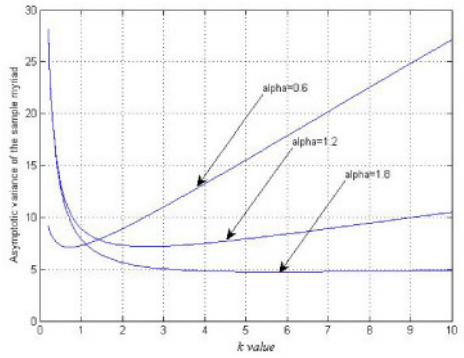

Fig. 1. Asymptotic variance of the myriad estimator for γ=2 and three fixed α

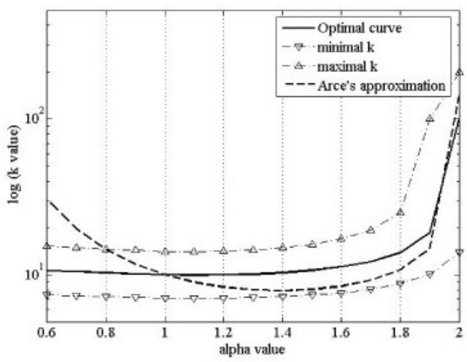

Fig. 2. $k_{opt\,A}(\alpha,\,\gamma)$ and $k_{opt}(\alpha,\,\gamma)$ for γ=10

The values k_{opt} have been obtained by analyzing 1996 values of k for each pair of α and γ. Fig. 1 shows the plots of the myriad asymptotic variance dependences upon k for fixed γ=2 and three different values of α, namely 0.6, 1.2, and 1.8.

The values k_{opt} obtained for γ equal to 0.3, 1 and 10 are also presented in Table 1. Besides, we determined the interval boundaries $[k_{\min}; k_{\max}]$ for which the variance of myriad estimator does not exceed minimally reachable σ^2_{Myr} by more than 3%. Such analysis has been done to demonstrate that the neighborhood of the dependence minimum is rather flat and there is always a range of $k \in [k_{\min}; k_{\max}]$ for which a loss of estimator accuracy can be considered as negligible.

Analysis of data presented in Table 1 shows that the interval $[k_{\min}; k_{\max}]$ becomes wider for greater α. Only a lower bound exists for Gaussian noise (α=2) - $k_{\min} \approx 3\sigma_G$.

Simultaneously, the obtained data indicate errors in $k_{opt\,A}(\alpha,\,\gamma)$ approximation for the cases when $\gamma \neq 1$. For comparison purpose, the curves $k_{opt\,A}(\alpha,\,\gamma)$ and $k_{opt}(\alpha,\,\gamma)$ are represented for γ=10 in Fig. 2. As it is seen, for α<1 the values $k_{opt\,A}$ start to increase and for α<0.8 they become out of the interval $[k_{\min}; k_{\max}]$. This is due to the first factor $\sqrt{\alpha/(2-\alpha)}$ in the expression for $k_{opt\,A}(\alpha,\,\gamma)$ which is incorrect. Thus, it is desirable to have an expression to correctly describe $k_{opt}(\alpha,\,\gamma)$ for any γ.

The paper [18] gives an approximation of $k_{opt}(\alpha,\,\gamma)$ that is better than $k_{opt\,A}(\alpha,\,\gamma)$ for γ=1. Thus, by using an approximation $k_{opt}(\alpha,\,\gamma)$ [18] as a basis and taking into account correct description of SαS data scale by $\gamma^{1/\alpha}$ we propose the following approximation for any γ:

$$k_{opt}^{appr}(\alpha,\gamma) = \left(-0.66 + 0.44e^{1.28\alpha} + \right.$$

$$\left. + \, 7.62 \cdot 10^{-34} e^{39.24\alpha}\right) \gamma^{1/\alpha}, \ \alpha \geq 0.3$$

$$\textit{and } k_{opt}^{appr}(\alpha,\,\gamma) = k_0, \text{ if } \alpha < 0.3, \qquad (14)$$

where k_0 is a reasonably small value, for example, 10^{-20}. To our best knowledge, there are no reported results concerning the noise and natural phenomena modeling using SαS distributions with $\alpha < 0.3$.

Fig. 3 gives examples of approximation curves $k_{opt}^{appr}(\alpha,\gamma)$ calculated according to (14) and empirically obtained $k_{opt}(\alpha,\,\gamma)$ for γ=0.3 and γ=8. The boundary curves $k_{\min}(\alpha,\,\gamma)$ and $k_{\max}(\alpha,\,\gamma)$ are also presented. As one can see, practically for all values of α there is a good coincidence of $k_{opt}^{appr}(\alpha,\gamma)$ and $k_{opt}(\alpha,\,\gamma)$. The only exception is the case of Gaussian noise when α=2. But even in this case $k_{opt}^{appr}(\alpha,\gamma)$ belongs to the interval $[k_{\min}; k_{\max}]$.

Hence, the myriad estimator with $k_{opt}^{appr}(\alpha,\gamma)$ is quasi-optimal for SαS distributions. Then, if α and γ are known a priori the tunable parameter value can be easily calculated from (14) where any α and γ can be used including γ considerably different from 1. Besides, analysis of the obtained values $k_{\min}$ and $k_{\max}$

TABLE I

THE VALUES k_{opt}, $k_{\min}$ AND $k_{\max}$ FOR THREE DIFFERENT γ AND $\alpha \in [0.6; 2]$

γ		A							
		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3
0.3	k_{opt}	0.08	0.08	0.13	0.21	0.30	0.41	0.54	0.69
	$k_{\min}$	0.08	0.08	0.09	0.15	0.22	0.30	0.39	0.49
	$k_{\max}$	0.08	0.09	0.18	0.29	0.42	0.58	0.77	0.98
1	k_{opt}	0.20	0.35	0.60	0.80	1.00	1.25	1.45	1.70
	$k_{\min}$	0.18	0.25	0.45	0.60	0.75	0.90	1.05	1.25
	$k_{\max}$	0.25	0.55	0.80	1.10	1.40	1.70	2.05	2.45
10	k_{opt}	10.6	10.5	10.3	10.1	10	9.95	10	10.10
	$k_{\min}$	7.40	7.35	7.30	7.20	7.10	7.05	7.05	7.15
	$k_{\max}$	15.3	14.9	14.6	14.3	14.1	14.1	14.2	14.45

γ		A						
		1.4	1.5	1.6	1.7	1.8	1.9	2
0.3	k_{opt}	0.85	1.05	1.29	1.62	2.15	3.59	∞
	$k_{\min}$	0.60	0.73	0.87	1.05	1.30	1.70	2.48
	$k_{\max}$	1.24	1.57	2.00	2.75	4.98	10	∞
1	k_{opt}	2.00	2.30	2.65	3.15	3.85	5.50	∞
	$k_{\min}$	1.40	1.60	1.85	2.10	2.45	3	4.40
	$k_{\max}$	2.85	3.35	4.00	5.00	7.00	100	∞
10	k_{opt}	10.35	10.70	11.25	12.15	13.85	18.50	∞
	$k_{\min}$	7.25	7.40	7.65	8.05	8.75	10.10	13.85
	$k_{\max}$	14.95	15.75	17.00	19.40	25.25	100	∞

shows that the selection of k within the interval $[k_{\min}; k_{\max}]$ (i.e., small errors in determination of k) does not result in considerable reduction of the myriad estimator accuracy.

III. MYRIAD ESTIMATOR ADAPTATION TO SαS PDF UNKNOWN PARAMETERS

In practice, the information concerning parameters of noise distribution is quite limited. Then the task is to adapt a processing method to a particular situation. If a sample myriad is used for non-Gaussian and, in particular, SαS PDF noise, such adaptation implies adaptive setting the tunable parameter [12, 14]. There are several approaches to solving this task [8], namely:

1. Estimation of α and γ with subsequent substitution of the obtained estimates in (14) (approach 1);
2. Determination of k_{opt} using some parameters that uniquely define statistics of the processes with SαS PDF (approach 2);
3. Application of alternative methods, for example, bootstrap [29, 30] (approach 3).

Below we consider the first two approaches keeping in mind that there are different methods for estimating α and γ. The third one requires considerably more computations [8, 30].

A. *Calculation of k_{opt} (approach 1)*

Note that parameter α characterizes the heaviness of SαS distribution tails. Independently from the distribution scale, the smaller α values correspond to heavier tails. Then the task is to find a statistical parameter for data sample able to uniquely characterize α. One such parameter is the percentile coefficient of kurtosis (PCK) [31] calculated as

$$P = \frac{Y_{75} - Y_{25}}{2\,(Y_{90} - Y_{10})}, \tag{15}$$

where Y_m denotes the m-th percentile of a data sample under consideration.

An advantage of PCK (15) for SαS distributions is that it is a monotonously increasing

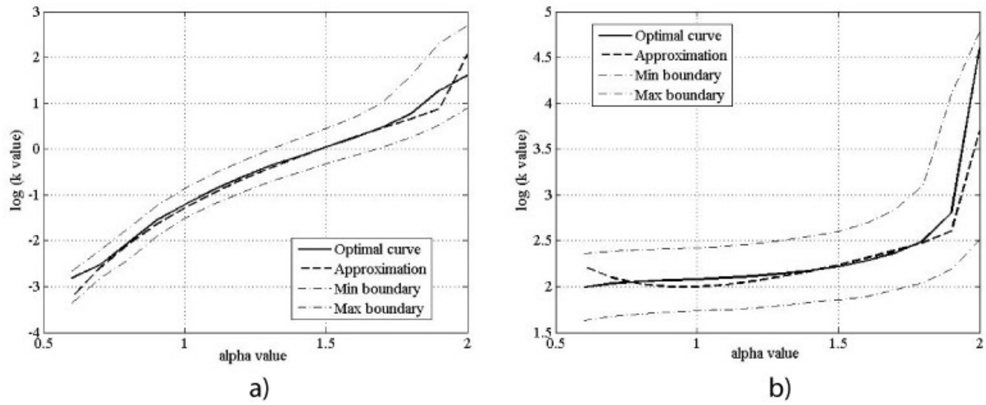

Fig. 3. Dependences $k_{opt}^{appr}(\alpha, \gamma)$ and $k_{opt}(\alpha, \gamma)$ for γ=0.3 (a) and γ=8 (b)

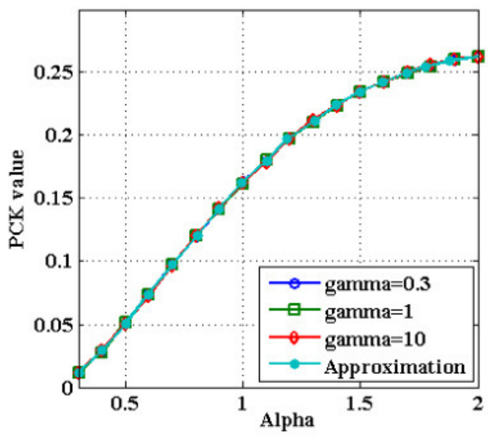

Fig. 4. Dependencies PCK(α) for three different values of γ and approximation (16) of the inverse function $f_1(P)$

function of α. Besides, for fixed α but different γ, ensemble averaged values of PCK are remain the same. This is clearly seen from plots presented in Fig. 4. Thus, for estimating α it is possible to use inverse function $\hat{\alpha} = f_1(P)$. For approximating $f_1(P)$, we propose to use the following 4-th order polynomial

$$\hat{\alpha} = 1.035 \cdot 10^3 P^4 - 419.8P^3 + 55.17P^2 + 2.051P + 0.286. \quad (16)$$

The approximating polynomial (16) is shown in Fig. 4. As it is seen, it describes the true function $f_1(P)$ well.

Table 2 presents true values of the parameter α for simulated data samples and the corresponding statistical characteristics (the mean $\langle\hat{\alpha}\rangle$ and standard deviation $\sigma_{\hat{\alpha}}$) for the estimates obtained by the considered method. Simulations have been performed for data samples of size 256 elements and for 1000 realizations for each α (γ=1). The results obtained show that the proposed method provides rather accurate estimates of α. The worst accuracy (the largest values of $\sigma_{\hat{\alpha}}$) is observed for large α. Certainly, accuracy is getting worse if a sample size is smaller and vice versa.

For estimating the scale of SαS distribution we propose to use the median of absolute deviations (MAD) [9, 31]:

$$\text{MAD} = \text{med}\{|y_n - \text{med}(y_n)|\}, \quad (17)$$

where med{...} denotes a sample median.

Behavior of the mean values of this parameter for different α and γ is shown in Fig. 5 (in logarithmic scale). Our investigations have demonstrated [8] that MAD (17) can be approximated as

$$\text{MAD} = C(\alpha) \cdot \gamma^{1/\alpha}, \quad (18)$$

where $C(\alpha)$ is the correcting factor that depends upon α and can be defined as:

$$C(\alpha) = 1.84\alpha^6 - 14.18\alpha^5 + 44.36\alpha^4 - 72.02\alpha^3 + 64.3\alpha^2 - 30.3\alpha + 7.02. \quad (19)$$

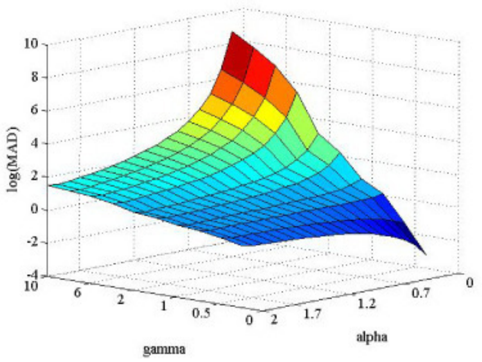

Fig. 5. Dependence log(MAD) on α and γ of SαS PDF

Then, as it follows from (18), after estimating $\hat{\alpha}$, $\gamma^{1/\alpha}$ can be calculated as

$$\gamma^{1/\alpha} = \mathrm{MAD}/C(\hat{\alpha}). \tag{20}$$

Then the first approach ($\hat{k}_{opt1}$) for k derivation is to substitute the obtained (16) and (20) values into expression (14). The accuracy of this approach will be analyzed in the next Section.

B. Other methods of α and γ estimation

It can be predicted that the performance of the adaptive myriad estimator described above depends upon accuracy of the estimates $\hat{\alpha}$ and $\hat{\gamma}$. Meanwhile, there are several algorithms for SαS parameters estimation from data sample. The method most similar to our technique described above was developed by McCulloch [24]. It is also based on percentiles. However, other percentiles (and, respectively, quantiles) are used in [24] than in (15). More exactly, the 95-th and 5-th percentiles are applied instead of the 90-th and 10-th. Several other methods were proposed by Ma and Nikias [25], Kogon and Williams [26], Tsihrintzis and Nikias [27]. These methods are based on different approaches and principles. In [25], the authors use the theory of fractional lower order moments. The method [26] is based on using the empirical characteristic function. The basic idea of the method [27] is to apply the theory of asymptotic extreme values and order statistics.

To compare the accuracy of the proposed method for α and γ estimation to the aforementioned methods, computer simulation using Monte-Carlo method has been carried out. In all cases, SαS distributions with location equal to zero were considered. M=1000 data samples have been generated for each sample size (N=100, 200, 500, 1000, 2000, and 5000) and for each pair of α and γ.

Note that for all of the considered estimation techniques we restricted the obtained α estimates. If they were greater than 2 we set $\hat{\alpha} = 2$. This peculiarity influences the method [27] for which the obtained standard deviation $\sigma_{\hat{\alpha}}$ is equal to 0 and $\langle\hat{\alpha}\rangle_M = 2$ for α=1.9 and 2 (see data in Figs. 6a and b for N=200 and 1000). Also note that the estimation method [24] can not be applied for estimation of α<0.5. Thus, the values $\langle\hat{\alpha}\rangle_M = 0.5$ and $\sigma_{\hat{\alpha}}$=0 for α<0.5 are explained by the fact that the estimated value $\hat{\nu}_\alpha$ was less than the smallest value tabulated in [24]. Due to this, we set in such situations. The obtained dependencies of $\langle\hat{\alpha}\rangle$ and $\sigma_{\hat{\alpha}}$ on α for γ=1 are presented for our method described in subsection 3.1 and the methods [24-27] in Figs. 6 and 7.

In general, all of the considered methods produce rather good and similar results. The method of McCulloch [24] provides slightly better accuracy than other methods for 1<α<1.7 whilst the method by Ma and Nikias [25] produces the smallest $\sigma_{\hat{\alpha}}$ for α<1. Our method is usually more accurate than the methods [26, 27] but less accurate than the methods [24, 25].

Since the estimates of α and γ are mutually dependent, less accurate estimation of α usually leads to a worse estimation of γ. This assumption has been verified for all considered methods for several fixed γ, different α and sample sizes. The main results show that our estimation method defined by expressions (17) and (20) as well as the methods [24] and [25] produce practically unbiased estimation of γ. $\hat{\gamma}$ obtained by the method [26] can be sufficiently biased if γ considerably differs from unity. Variances of the obtained estimates were the smallest for our method defined by (17) and (20) and the method [24].

These results are in agreement with conclusions drawn in the paper [32]. Thus, it is worth paying more attention to the latter two meth-

TABLE II
Statistical characteristics of estimates $\hat{\alpha}$ for the first approach

α	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
$\langle\hat{\alpha}\rangle$	0.317	0.384	0.487	0.602	0.710	0.813	0.905	0.999	1.094
$\sigma_{\hat{\alpha}}$	0.017	0.041	0.062	0.077	0.082	0.081	0.084	0.095	0.111
α	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\langle\hat{\alpha}\rangle$	1.199	1.311	1.423	1.555	1.655	1.766	1.858	1.950	1.999
$\sigma_{\hat{\alpha}}$	0.142	0.166	0.190	0.244	0.268	0.293	0.326	0.342	0.356

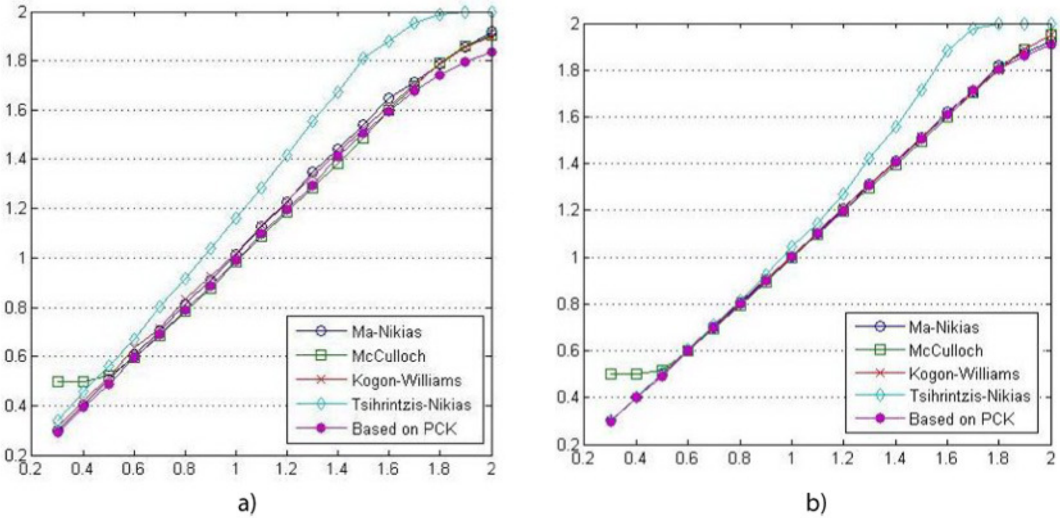

Fig. 6. Dependences of $\langle\hat{\alpha}\rangle_M$ upon true α for the considered methods for γ=1, N=200 (a) and 1000 (b)

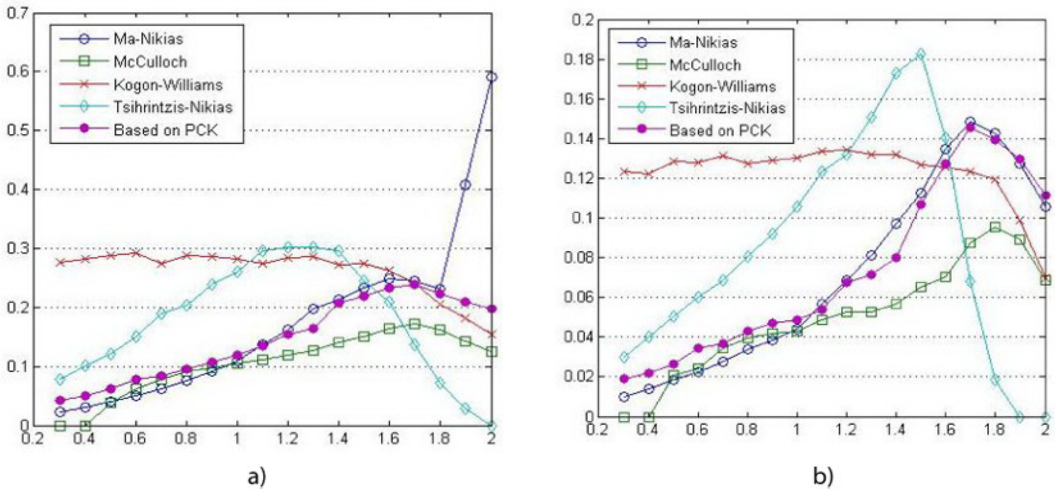

Fig. 7. Dependences of $\sigma_{\hat{\alpha}}$ upon true α for the considered methods for γ=1, N=200 (a) and 1000 (b)

ods in further analysis (see Section 4).

C. Estimation of k_{opt} using parameters that uniquely describe processes with SαS PDF (approach 2)

Now consider the approach 2 that presumes that can be obtained without using (14). Since PCK and MAD together correctly describe processes with SαS PDF, we propose to determine $\hat{k}_{opt2}$ as

$$\hat{k}_{opt2} = \mathrm{MAD} \cdot f(P), \qquad (21)$$

where the function $f(P)$ can be determined from k_{opt} obtained earlier (see Section 2) and $\langle \mathrm{MAD} \rangle_M$ (see subsection 3.1) as

$$f(P) = k_{opt} / \langle \mathrm{MAD} \rangle_M . \qquad (22)$$

The values of the function $f(P)$ have been determined for a wide range of α and γ. To make the expression (21) practically applicable to all possible situations it was necessary to find a good approximation of $f(P)$. The following approximation has been proposed:

$$f(P) = 3.029 \cdot 10^5 P^6 - 2.011 \cdot 10^5 P^5 + 4.95 \cdot 10^4 P^4 - 5434 P^3 + 265.8 P^2 + 2.36 P + 0.032, \quad \text{for } P \in (0; 0.25]$$

$$f(P) = 2.53 \cdot 10^5 P^2 - 1.301 \cdot 10^5 P + 1.673 \cdot 10^4, \quad \text{for } P \in (0.25; +\infty] \qquad (23)$$

Two polynomials for describing $f(P)$ are used due to the fact that one polynomial was unable to produce appropriate accuracy. Note that PCK is larger than 0.25 only for distributions close to Gaussian. The analysis of the approach 2 accuracy will be carried out in the next Section.

IV. Accuracy analysis for the proposed adaptive myriad estimators

Obviously, one can analyze both the accuracy of adaptive estimation of $\hat{k}_{opt}$ and accuracy of location parameter estimation by means of the adaptive myriad. In this Section we follow the latter way. The variances of location estimates $V_{Myr}\left(\hat{k}_{opt1}\right)$ (for the approach 1) and $V_{Myr}\left(\hat{k}_{opt2}\right)$ (for the approach 2) obtained for the myriad estimators that use $\hat{k}_{opt1}$ and $\hat{k}_{opt2}$, respectively, have been calculated by numerical simulations. Several pairs of the parameters α and γ for the large number of realizations $\grave{I}$=2000 and the sample size N=256 have been investigated. These results are presented in Table 3 for three values of γ equal to 0.3, 1 and 10. Besides, the variance values for the myriad estimator with properly set k ($V_{Myr}(k_{opt})$) have been calculated as well as the theoretically reachable variance ($V_{Myr}^{theor} = \sigma^2_{Myr}(\alpha,\gamma)/N$).

Analysis of data in Table 3 shows that almost always V_{Myr}^{theor} is slightly smaller than $V_{Myr}(k_{opt})$. For $\alpha \leq 1.3$ the values $V_{Myr}\left(\hat{k}_{opt1}\right)$and $V_{Myr}\left(\hat{k}_{opt2}\right)$ are slightly larger than V_{Myr}^{theor}, the difference is few percent. For $\alpha \in [1.4; 2]$ the difference is larger, up to 50% (consider the case α=1.7). Such a situation deals with not very accurate estimation of $\hat{\alpha}$ for such α (see data in Table 2) due to small slope of the characteristic in Fig. 4, i.e., small variations in estimated PCK can lead to considerable variations of the estimates $\hat{\alpha}$.

Comparison of both proposed approaches for adaptive selection of k_{opt} shows that variances $V_{Myr}\left(\hat{k}_{opt1}\right)$and $V_{Myr}\left(\hat{k}_{opt2}\right)$ do not differ a lot. Thus, in general, both methods can be used in practice. Computation expenses for them are practically the same since most computations are due to data sorting.

We have also tested the method of McCulloch [24] for estimating α and γ at the first stage of the approach 1. This modification has produced slightly smaller variance $V_{Myr}\left(\hat{k}_{opt3}\right)$ in comparison to $V_{Myr}\left(\hat{k}_{opt1}\right)$and $V_{Myr}\left(\hat{k}_{opt2}\right)$ for α about 1.6. For other α, the values $V_{Myr}\left(\hat{k}_{opt3}\right)$, $V_{Myr}\left(\hat{k}_{opt1}\right)$and $V_{Myr}\left(\hat{k}_{opt2}\right)$ are almost the same. Therefore, it is reasonable to apply the approach 1 equipped with McCulloch's estimation methods [24] used at the first stage. One motivation in favor of this recommendation is

TABLE III

VARIANCES OF THE MYRIAD ESTIMATORS WITH k_{opt}, $\hat{k}_{opt1}$, $\hat{k}_{opt2}$ AND $\hat{k}_{opt3}$ FOR γ EQUAL TO 0.3, 1 AND 10

		α						
		0.6	0.7	0.8	0.9	1	1.1	1.2
$\gamma = 0.3$	V_{Myr}^{theor}	0.000052	0.000136	0.000286	0.000477	0.000702	0.000944	0.001187
	$V_{Myr}(k_{opt})$	0.000058	0.000144	0.000307	0.000492	0.000737	0.000986	0.001255
	$V_{Myr}(\hat{k}_{opt1})$	0.000059	0.000159	0.000311	0.000456	0.000698	0.000969	0.001200
	$V_{Myr}(\hat{k}_{opt2})$	0.000058	0.000159	0.000295	0.000512	0.000731	0.000977	0.001148
	$V_{Myr}(\hat{k}_{opt3})$	0.000055	0.000147	0.000304	0.000488	0.000734	0.001011	0.001278
	V_{Med}	0.000083	0.000211	0.000371	0.000670	0.000851	0.001092	0.001551
$\gamma = 1$	V_{Myr}^{theor}	0.00277	0.00443	0.00585	0.00696	0.00781	0.00843	0.00884
	$V_{Myr}(k_{opt})$	0.00288	0.00470	0.00608	0.00733	0.00787	0.00883	0.00945
	$V_{Myr}(\hat{k}_{opt1})$	0.00320	0.00449	0.00579	0.00702	0.00782	0.00902	0.00903
	$V_{Myr}(\hat{k}_{opt2})$	0.00321	0.00480	0.00606	0.00780	0.00783	0.00907	0.00932
	$V_{Myr}(\hat{k}_{opt3})$	0.00303	0.00467	0.00624	0.00738	0.00918	0.00858	0.00963
	V_{Med}	0.00445	0.00636	0.00726	0.00860	0.01028	0.00969	0.01160
$\gamma = 10$	V_{Myr}^{theor}	5.83	3.09	1.81	1.150	0.781	0.554	0.410
	$V_{Myr}(k_{opt})$	6.19	3.43	1.90	1.161	0.849	0.568	0.417
	$V_{Myr}(\hat{k}_{opt1})$	6.32	3.52	1.92	1.112	0.797	0.600	0.405
	$V_{Myr}(\hat{k}_{opt2})$	7.29	3.56	1.92	1.291	0.792	0.572	0.414
	$V_{Myr}(\hat{k}_{opt3})$	6.67	3.47	1.89	1.126	0.788	0.542	0.390
	V_{Med}	10.12	4.72	2.55	1.448	0.949	0.732	0.485

that the values $\alpha \approx 1.6$ are more often met in practical situations than others especially smaller ones [5].

It can be also interesting to compare the accuracy of the proposed adaptive myriad estimators to the accuracy provided by some non-adaptive estimator. For this purpose, we have obtained the values of variance V_{Med} for the standard median estimator and presented them in Table 3 for all considered pairs of γ and α. As one can see, the proposed adaptive estimators almost always provide better accuracy than the standard median estimator (except cases α=1.6 and α=1.7 for the $\hat{k}_{opt1}$ and $\hat{k}_{opt2}$ algorithms). The smallest difference (few percent) is observed for α about 1.5, but the difference is considerable for small α.

Then, the following question arises: under what conditions (for what minimal value N) is it reasonable to use adaptive myriad estimator instead of some simpler estimator, e.g., the sample median that does not require any adaptation? To answer this question, we have to carry out simulations for N smaller than that one used for obtaining data in Table 3 (N=256). As expected, the tendency to increase the variances $V_{Myr}\left(\hat{k}_{opt1}\right)$, $V_{Myr}\left(\hat{k}_{opt2}\right)$ and $V_{Myr}\left(\hat{k}_{opt3}\right)$ was observed. Moreover, ratios $V_{Myr}\left(\hat{k}_{opt1}\right)/V_{Med}$, $V_{Myr}\left(\hat{k}_{opt2}\right)/V_{Med}$ and $V_{Myr}\left(\hat{k}_{opt3}\right)/V_{Med}$ increase if sample size N becomes smaller. Numerical simulation results obtained for N=64 are presented as plots of $V_{Myr}\left(\hat{k}_{opt1}\right)$, $V_{Myr}\left(\hat{k}_{opt2}\right)$, $V_{Myr}\left(\hat{k}_{opt3}\right)$ and V_{Med} versus α for fixed γ=1 in Fig. 8.

As one can see, for a wide range of α the dif-

TABLE III (CONTINUED). VARIANCES OF THE MYRIAD ESTIMATORS WITH k_{opt}, $\hat{k}_{opt1}$, $\hat{k}_{opt2}$ AND $\hat{k}_{opt3}$ FOR Γ EQUAL TO 0.3, 1 AND 10

		α							
		1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
γ = 0.3	V_{Myr}^{theor}	0.00142	0.00164	0.00183	0.00200	0.00214	0.00224	0.00230	0.00232
	$V_{Myr}(k_{opt})$	0.00146	0.00167	0.00185	0.00206	0.00236	0.00227	0.00245	0.00251
	$V_{Myr}(\hat{k}_{opt1})$	0.00154	0.00195	0.00235	0.00250	0.00279	0.00323	0.00260	0.00240
	$V_{Myr}(\hat{k}_{opt2})$	0.00156	0.00190	0.00217	0.00263	0.00279	0.00302	0.00259	0.00244
	$V_{Myr}(\hat{k}_{opt3})$	0.00149	0.00160	0.00178	0.00217	0.00228	0.00264	0.00253	0.00233
	V_{Med}	0.00160	0.00213	0.00234	0.00276	0.00311	0.00316	0.00315	0.00384
γ = 1	V_{Myr}^{theor}	0.00907	0.00917	0.00916	0.00905	0.00887	0.00861	0.00826	0.00777
	$V_{Myr}(k_{opt})$	0.00982	0.00949	0.00920	0.00921	0.00900	0.00870	0.00893	0.00779
	$V_{Myr}(\hat{k}_{opt1})$	0.00921	0.01005	0.01068	0.01154	0.01306	0.01065	0.00909	0.00792
	$V_{Myr}(\hat{k}_{opt2})$	0.01018	0.01019	0.01150	0.01246	0.01448	0.00994	0.00980	0.00776
	$V_{Myr}(\hat{k}_{opt3})$	0.00961	0.00944	0.00971	0.00978	0.00910	0.01062	0.00923	0.00859
	V_{Med}	0.01112	0.01242	0.01205	0.01166	0.01242	0.01209	0.01214	0.01217
γ = 10	V_{Myr}^{theor}	0.313	0.246	0.197	0.161	0.133	0.111	0.093	0.077
	$V_{Myr}(k_{opt})$	0.321	0.258	0.204	0.161	0.142	0.111	0.094	0.080
	$V_{Myr}(\hat{k}_{opt1})$	0.307	0.301	0.233	0.207	0.177	0.151	0.111	0.077
	$V_{Myr}(\hat{k}_{opt2})$	0.334	0.274	0.240	0.228	0.213	0.161	0.107	0.0812
	$V_{Myr}(\hat{k}_{opt3})$	0.336	0.263	0.219	0.169	0.155	0.133	0.105	0.0805
	V_{Med}	0.406	0.322	0.244	0.216	0.184	0.160	0.137	0.125

ference between $V_{Myr}\left(\hat{k}_{opt1}\right)$, $V_{Myr}\left(\hat{k}_{opt2}\right)$ and V_{Med} is practically insufficient. Only for $\alpha>1.8$ and $\alpha<1.1$ it occurs reasonable to apply adaptive robust estimator. Therefore, from practical viewpoint, there is no reason to apply the proposed adaptive myriad estimators with $\hat{k}_{opt1}$ and $\hat{k}_{opt2}$ instead of simpler sample median for data samples with N smaller than 70. Meanwhile the adaptive myriad with $\hat{k}_{opt3}$ produces the best results among considered estimators. The benefit in comparison to the sample median is not less than 10%.

V. Application of adaptive myriad estimators in RDFT based signal processing

RDFT is a tool that can be effectively used for analyzing and filtering 1D signals embedded in non-Gaussian noise [13, 14]. A basic idea is that in standard DFT

$$X_S(i) = (1/N)\sum_{n=1}^{N} y(n)\exp\left(-j2\pi in/N\right) =$$
$$= mean\left\{Re\left(y(n)\exp\left(-j2\pi in/N\right)\right)\right\} +$$
$$+jmean\left\{Im\left(y(n)\exp\left(-j2\pi in/N\right)\right)\right\}, \tag{24}$$
$$i = \overline{0, N-1},$$

the operation of mean calculation is replaced by some robust estimation $T\{...\}$. Then one obtains

$$X_R(i) = T\left\{Re\left(y(n)\exp\left(-j2\pi in/N\right)\right)\right\}$$
$$+jT\left\{Im\left(y(n)\exp\left(-j2\pi in/N\right)\right)\right\} \tag{25}$$

or, if an input process is real-valued,

$$X_R(i) = T\left\{y(n)\cos\left(2\pi in/N\right)\right\}$$
$$-jT\left\{y(n)\sin\left(2\pi in/N\right)\right\}. \tag{26}$$

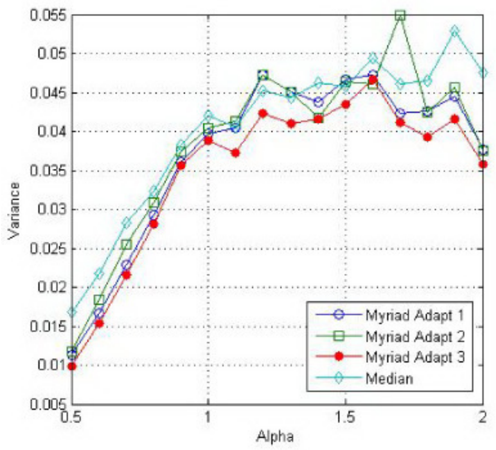

Fig. 8. Variance dependencies of the proposed estimators and sample median for SaS noise with γ=1 and different α ("Myriad Adapt metricconverterProductID1"1" – adaptive k evaluation by means of approach 1; "Myriad Adapt metricconverterProductID2"2" - adaptive k calculation by approach 2; "Myriad Adapt metricconverterProductID3"3" - adaptive k evaluation by means of approach 1 using McCulloch estimators)

RDFT can be used for estimation of signal spectrum as $X_R^p(i) = |X_R(i)|^2$ or and its further analysis. In the latter case, $X_R(i)$ is multiplied by a desired spectral characteristic $\dot{K}(j\omega)$ represented in discrete form as $\dot{K}(i)$ and then standard inverse DFT is applied [13, 14].

Several different RDFT forms that employ different estimators, both non-adaptive [13] and adaptive [14], have been proposed and analyzed. Our goal below is to provide brief performance analysis for the proposed adaptive myriad estimators used in RDFT framework.

Consider a case of filtering a frequency modulated (FM) signal embedded into non-Gaussian (impulsive) noise with the primary purpose to remove impulses. As it is known, the standard DFT is sensitive to spikes and RDFT denoising is able to carry out the considered task better [13]. Thus, we have simulated a test FM signal as

$$s(n) = A\cos\left(2\pi n\left[an+b\right]\right), \qquad (27)$$

where A=5 is the amplitude value, a and b are equal to 17.5 and 5 that corresponds to the LFM-signal with initial frequency F_1=5 Hz and final frequency F_2=40 Hz.

Then the SαS noise with several different pairs of α and γ has been added and spike removal by applying different robust estimators $T\{...\}$ according to (25), using $\dot{K}(i) = 1$ and performing inverse DFT [13] has been carried out. Efficiency of such denoising can be characterized by output MSE (MSE_{out}) values for different robust estimators compared between each other and with respect to input MSE (MSE_{inp}). The obtained results are presented in Tables 4 and 5 for two different γ and three different α. AM1 means adaptive myriad for the approach 1 that uses the expressions (16) and (20). AM2 refers to the approach 2 described in subsection 3.3. AM3 is the adaptive myriad for the approach 1 based on McCulloch estimators.

As one can see, if α=1.2 or α=1.5 MSE_{out} is considerably decreased in comparison to MSE_{inp} due to applying RDFT based filtering for all four robust estimators, namely, standard median and three proposed adaptive myriad estimators. For the myriad form of RDFT denosing, there is an improvement in comparison to the median form RDFT or all considered RDFT based denoising techniques provide approximately the same MSE_{out}. Comparing four adaptive myriad estimators, our recommendation is to use RDFT based on AM2. The median based RDFT does not perform well for α=1.8.

Note that α from 1.2 to 1.8 may correspond to atmospheric noise that influences input blocks of radar and communication systems [33].

VI. Conclusions

This paper presents theoretically optimal values of the myriad estimator tunable parameter for $\gamma \neq 1$. Besides, the boundary values of k for which losses in accuracy of the estimator can be considered negligible are given. The approximation formula obtained in [18] is generalized for arbitrary γ and comparison of $k_{opt}^{appr}(\alpha,\gamma)$ with theoretical optimal values is done.

Two approaches for adaptation of myriad estimator to a priori unknown parameters of SαS

TABLE IV
Output MSE values for RDFT based signal filtering

Data processing method	γ=0.5		
	α=1.2	α=1.5	α=1.8
Standard DFT	7.26	2.53	1.44
RDFT based on AM1	2.08	1.63	1.29
RDFT based on AM2	2.07	1.58	1.28
RDFT based on AM3	3.01	1.84	1.35
Median based RDFT	2.32	2.22	2.12

TABLE V
Output MSE values for RDFT based signal filtering

Data processing method	γ=1		
	α=1.2	α=1.5	α=1.8
Standard DFT	23.53	4.87	2.09
RDFT based on AM1	2.55	2.17	1.78
RDFT based on AM2	2.57	2.11	1.69
RDFT based on AM3	2.54	2.15	1.64
Median based RDFT	2.55	2.39	2.26

distribution are proposed. The first method and the corresponding algorithms presume estimation of the parameters α and γ for a processed data sample. Then the approximation formula for determination of k_{opt} for the myriad estimator to be used for location estimation is applied. The second approach is based on direct calculation of the tunable parameter by using the estimates of PCK and MAD for an observed data sample.

Performance analysis of the proposed adaptive myriad estimators has demonstrated that they provide rather high accuracy for location estimation of SαS PDF data samples. Variance of such estimates is greater than theoretically reachable limit by no more than 50%. The application of the myriad estimator with the proposed adaptation procedures $\hat{k}_{opt1}$ and $\hat{k}_{opt2}$ is reasonable for sample size larger than 70. For smaller sample sizes it is recommended to apply McCulloch estimators within a framework of approach metricconverterProductID1. In1. In practice, main benefits of the proposed adaptive estimators are observed if heaviness of noise PDF tails increases.

Our future work will focus on performance analysis of the designed adaptive myriad estimator for other (not only SαS) non-Gaussian heavy tail distributions.

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