Estimation of the sinusoidal signal frequency based on the marginal median DFT

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Abstract— **The marginal-median DFT is used for estimation of complex sinusoidal signals embedded in an impulse noise environment. Expression for the marginal-median DFT of the sinusoidal signal in the neighborhood of the exact frequency is derived. Two specific displacement techniques are proposed in order to achieve an accurate estimation of frequency displaced from the frequency grid. They are based on specific ratio of the marginal-median DFT magnitudes for samples in the neighborhood of the marginal-median DFT maximum calculated over the frequency grid. Efficiency and accuracy of the proposed techniques is proved for mixed Gaussian and impulse noise environment.**

I. Introduction

Numerous interpolation schemes in the Fourier domain have been proposed for precise frequency estimation of the sinusoidal signals. Displacement based techniques have attracted significant attention due to simple and efficient realizations [1], [2], [3]. This group of techniques is based on the magnitudes of the discrete Fourier transform (DFT) maximum, as well as on the two adjacent samples. The technique proposed by Quinn is one of the most commonly used. It is applied on the DFT with rectangular window function. Alternative techniques are proposed for windowed DFT forms [3]. Recently, a very accurate iterative procedure has been proposed for the frequency displacement estimation in [4]. It has been shown that this technique produces variance in the frequency estimation approaching to just about 1.5% higher value than the Cramér-Rao lower bound (CRLB) for the Gaussian noise environment.

The standard DFT is a maximum likelihood estimation of the signal spectra for Gaussian noise environment. Unfortunately, the DFT, like other linear techniques, exhibits low accuracy for signals corrupted by impulse noises. The robust DFT forms have been proposed recently in order to handle the spectral analysis issue for signals corrupted by impulse noise [5], [6]. These transforms are derived according to robust statistics concept introduced by Huber [7]. However, methods for accurate interpolation of robust DFT coefficients (for accurate estimation of signal frequency) are not studied yet, and this paper is the first step in this direction.

Here we consider estimation of sinusoidal signals by using the marginal-median robust DFT form. The first task is to derive exact values for the marginal-median DFT for single-tone (continuous-time) signal around the spectral peak. It is derived by using continuous-time median function according to [8], [9]. It will be shown that that this expression is an accurate approximation for median of discretized signals used in numerical evaluation. As opposed to the standard DFT case, magnitude of the marginal-median DFT for a single-tone signal, calculated over frequency grid, depends on a signal phase. It makes estimation of the signal parameters based on the robust DFT forms a more challenging task.

In this paper we consider two approaches for accurate spectral estimation based on the robust DFT form. Proposed techniques are developed by using a specific ratio of the robust DFT values displaced from the robust DFT maximum for half of the sampling grid distance in the frequency domain. Motivation for using this ratio is found in the similar measure used in [4] for the standard DFT. Deter-

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mined ratio, $h(\delta,\varphi)$, is function of δ (displacement) and φ (signal phase). The first proposed technique for displacement estimation is calculation of the inverse function $\hat{\delta} = h^{-1}(\delta,\hat{\varphi})$. However, this inverse evaluation requires that the signal phase be estimated in advance. The second technique is based on the iterative procedure from [4]. Difference between the proposed one and the technique from [4] is in the applied DFT form. Note that estimation of the signal phase is not required in the iterative technique. The iterative procedure starts with a coarse estimate determined based on the position of the robust DFT maximum calculated over the frequency grid. Subsequent iterations are determined by evaluation of the ratio $h(\delta,\varphi)$. Important property of $h(\delta,\varphi)$ is the fact that $|h(\delta,\varphi)| \leq |\delta|$. It means that the proposed procedure converges toward the true displacement value. Expected drawback of the iterative procedure is calculation complexity that depends on the number of iterations in the procedure. However, it will be shown in numerical study that the required number of iterations is small.

The paper is organized as follows. Brief overview of interpolation techniques for the standard DFT case that are related to our approach is presented in Section II. Values of the marginal-median DFT in the neighborhood of the true signal frequency are analyzed in Section III. Proposed techniques for interpolation of the marginal-median DFT are described in Section IV. Numerical analysis has been presented in Section V. Conclusion with further steps in this research are given in Section VI.

II. Interpolation of the standard DFT

Our goal is to estimate frequency (and other parameters) of complex sinusoid $f(t) = A\exp(j\omega_0 t + j\varphi)$, embedded in a white noise $\nu(t)$, $x(t) = f(t) + \nu(t)$, based on discretized observations $x(n) = x(n\Delta t)$, where Δt is the sampling rate. We assume that we have N samples of the signal within the interval of interest $T = N\Delta t$. Without loss of generality, even number of samples is assumed.

For the Gaussian noise environment, the coarse frequency estimation can be performed by using the standard DFT:

$$X(k) = \frac{1}{N}\sum_{n=-N/2}^{N/2-1} x(n)W_N^{nk} =$$

$$= \text{mean}\{x(n)W_N^{nk} \mid n \in [-N/2, N/2)\},$$
$$k \in [-N/2, N/2), \qquad (1)$$

where $W_N = \exp(-j2\pi/N)$. Numbers $k \in [-N/2, N/2)$ in the DFT correspond to the analog frequencies $\omega = 2\pi k/T$. For $\omega_0 = 2\pi l/T$, where $l \in [-N/2, N/2)$, the DFT exhibits:

$$X(k) = \begin{cases} A\exp(j\varphi) & k = l \\ 0 & \text{elsewhere.} \end{cases} \qquad (2)$$

However, for frequency displaced from the frequency grid $\omega_0 = 2\pi l/T + \delta$, where $\delta \in [-\pi/T, \pi/T]$, the DFT exhibits:

$$X(l) = \frac{A}{N}\sum_{n=-N/2}^{N/2-1} \exp(j\delta n\Delta t + j\varphi)$$

$$= \frac{A\exp(j\varphi)}{N}\exp(-j\delta T/2)\frac{1-\exp(j\delta T)}{1-\exp(j\delta\Delta t)}$$

$$= \frac{A\exp(-j\delta T/2 - j\varphi)}{N}\frac{\sin\delta T/2}{\sin\delta\Delta t/2}. \qquad (3)$$

It can be seen that the maximum of the standard DFT magnitude is achieved at $k = l$ that is the closest to the true frequency. The important issue is that the magnitude of the DFT is not phase dependent. The coarse frequency estimation can be performed based on the position of the DFT maximum:

$$\hat{\omega}_0 = \frac{2\pi}{T}\hat{k}_0 = \frac{2\pi}{T}\arg\max_k |X(k)|. \qquad (4)$$

For a fine estimation it is required to estimate displacement δ by using some available interpolation technique. One idea is to evaluate DFT over a denser frequency grid, but it would substantially increase the calculation complexity. Quinn has proposed in [1] an algorithm for estimation of displacement based on the DFT maximum $X(\hat{k}_0)$ and adjacent DFT samples $X(\hat{k}_0 \pm 1)$. That algorithm is widely used due to its simplicity. Single step formulas for evaluation of the displacement in the case of the

DFT windowed with smooth window functions are also used in practice [3].

Recently a simple iterative scheme has been proposed in [4]. After coarse estimation in (4), new frequency is evaluated based on the DFT magnitude for $\hat{k}_0 \pm 1/2$. Next iterations are repeated for the updated frequency. It has been shown that this algorithm achieves excellent accuracy for signals embedded in the white Gaussian noise producing unbiased estimate with variance that is just above 1.5% higher than the asymptotic CRLB [2], [4]:

$$CRLB = \frac{6\sigma^2}{A^2 N(N^2-1)(\Delta t)^2}, \tag{5}$$

where σ^2 is the variance of the Gaussian white noise.

III. Robust DFT around frequency of the sinusoidal signal

The standard DFT, like other commonly used linear techniques, is sensitive to the impulse noise influence. The robust DFT forms are introduced to produce accurate spectral estimation of non-noisy signal's DFT for signals corrupted by an impulse noise [5], [6]. The robust DFT forms are used for filtering of high-pass signals corrupted by a significant impulse noise amount, as well as for parametric estimation of signals embedded in impulse noise [10], [11]. The marginal-median form of the robust DFT is considered here. It is given as:

$$\begin{aligned} X_M(k) = \text{median}\{&\text{Re}\{x(n)W_N^{nk}\}| \\ &n \in [-N/2, N/2)\} \\ +j\text{median}\{&\text{Im}\{x(n)W_N^{nk}\}| \\ &n \in [-N/2, N/2)\}, \end{aligned} \tag{6}$$

for $k \in [-N/2, N/2)$. Note that evaluation of the marginal-median DFT is more consuming than that of the standard DFT. It leads to simple conclusion that evaluation of the marginal-median over denser frequency grid is even more unacceptable than a similar procedure in the case of the standard DFT. Then, an interpolation technique for precise frequency estimation is even more important in the case of the marginal-median DFT. For complex sinusoid with frequency on the grid $\omega = 2\pi k/T$, $k \in [-N/2, N/2)$, the marginal-median DFT form produces the same results as the standard DFT (2). However, for signals that are dislocated from the frequency grid, an expression for the robust DFT values is not derived yet.

In order to give general formula for the marginal-median DFT of the complex sinusoidal, the continuous-time signal $f(t) = A\exp(j\omega_0 t + j\varphi)$ within $t \in [-T/2, T/2]$ is considered. The marginal-median DFT of the continuous-time signal can be written as:

$$\begin{aligned} \tilde{X}_M(\omega) = \text{median}\{&\text{Re}\{x(t)\exp(-j\omega t)\}| \\ &t \in [-T/2, T/2]\} \\ +j\text{median}\{&\text{Im}\{x(t)\exp(-j\omega t)\}| \\ &t \in [-T/2, T/2]\}. \end{aligned} \tag{7}$$

The continuous-time median for the considered interval is a value of argument function for which half of the interval produces higher and the other half gives smaller values of the argument function [8],[9]. Assume that frequency of the sinusoid is displaced from the frequency grid, $\omega_0 = 2\pi k_0/T + \delta$. Then, the marginal-median DFT calculated for frequency on the frequency grid can be denoted as

$$\begin{aligned} \tilde{X}_M(2\pi k_0/T) &= X(\delta, \varphi) \\ &= A[R(\delta,\varphi) + jI(\delta,\varphi)], \end{aligned} \tag{8}$$

since it depends on δ and φ. After tedious but straightforward derivations we have found that

$$R(\delta,\varphi) = $$

$$= \begin{cases} \cos\varphi, \\ \quad \delta \in [-\frac{4|\varphi|}{T}, \frac{4|\varphi|}{T}] \\ \cos\frac{\delta T}{4}, \\ \quad \delta \in [-\frac{8\pi}{3T} + \frac{4|\varphi|}{3T}, -\frac{4|\varphi|}{T}] \cup \\ \quad \cup [\frac{4|\varphi|}{T}, \frac{8\pi}{3T} - \frac{4|\varphi|}{3T}] \\ \cos(\frac{2\pi}{3} - \frac{|\varphi|}{3}), \\ \quad \delta \in [\frac{-8\pi-8|\varphi|}{3T}, \frac{-8\pi+4|\varphi|}{3T}] \cup \\ \quad \cup [\frac{8\pi-4|\varphi|}{3T}, \frac{8\pi+8|\varphi|}{3T}] \\ -\cos(\frac{\delta T}{8}), \\ \quad \delta \in [-\frac{3\pi}{T}, -\frac{8\pi}{3T} - \frac{8|\varphi|}{3T}] \cup \\ \quad \cup [\frac{8\pi}{3T} + \frac{8|\varphi|}{3T}, \frac{3\pi}{T}], \end{cases} \tag{9}$$

$$I(\delta,\varphi) = $$

$$= \begin{cases} \sin\varphi, \\ \quad \delta \in [-\frac{4(\pi/2-|\varphi|)}{T}, \frac{4(\pi/2-|\varphi|)}{T}] \\ \sin\frac{\delta T}{4}, \\ \quad \delta \in [\frac{-8\pi+4(\pi/2-|\varphi|)}{3T}, \frac{-4(\pi/2-|\varphi|)}{T}] \cup \\ \quad \cup [\frac{4(\pi/2-|\varphi|)}{T}, \frac{8\pi-4(\pi/2-|\varphi|)}{3T}] \\ \sin(\frac{2\pi}{3} - \frac{|\varphi|}{3}), \\ \quad \delta \in [\frac{-8\pi-8(\pi/2-|\varphi|)}{3T}, \frac{-8\pi+4(\pi/2-|\varphi|)}{3T}] \cup \\ \quad \cup [\frac{8\pi-4(\pi/2-|\varphi|)}{3T}, \frac{8\pi+8(\pi/2-|\varphi|)}{3T}] \\ -\sin(\frac{\delta T}{8}), \\ \quad \delta \in [\frac{-3\pi}{T}, \frac{-8\pi-8(\pi/2-|\varphi|)}{3T}] \cup \\ \quad \cup [\frac{8\pi+8|\varphi|}{3T}, \frac{3\pi}{T}], \end{cases}$$

$$I(\delta,\varphi) = \begin{cases} -R(\delta, -\pi/2-\varphi) & \varphi \le 0 \\ R(\delta, \pi/2-\varphi) & \varphi \ge 0. \end{cases} \tag{10}$$

Note that $X_M(k)$ is different from $X(\delta,\varphi)$ since $X_M(k)$ is evaluated by using a discrete set of samples. Real parts of $X(\delta,\varphi)$ and $X_M(k)$, for $N = 32$, $N = 128$, $N = 1024$ for $\varphi = 0$, $\varphi = \pi/4$, are visualized in Fig. 1. Difference between $X(\delta,\varphi)$ and $X_M(k)$ is small even for $N = 32$, especially in the main lobe (for $|\delta| \le \pi/T$) and two neighbor lobes ($3\pi/T \ge |\delta| > \pi/T$). It can be concluded, based on Fig.1, that expressions (9)-(10) may be used as an accurate approximation of $X_M(\omega)$ in the interval around detected spectral maximum.

IV. Algorithms for accurate frequency estimation

Based on the previous analysis, we propose two algorithms for accurate robust DFT interpolation. A ratio of the robust DFT samples around maximum detected in the coarse phase will be analyzed in Section IV.A. Two techniques based on this ratio are proposed in Section IV.B: the iterative procedure motivated by Aboutanios's paper and the single step approach.

A. Ratio $h(\delta,\varphi)$

We propose two techniques for estimation of the sinusoidal signals parameters. These techniques are motivated with similar development in the case of the standard DFT and Gaussian noise environment. The main ingredient in these procedures is the specific ratio between magnitudes of the marginal-median DFT for two frequencies displaced for $\pm\Delta\omega/2 = \pm\pi/T$ (half of the frequency sampling interval) from the position of the robust DFT maximum $\hat{\omega} = \hat{k}_0\Delta\omega$, where $\hat{k}_0$ is a coarse frequency estimate calculated over the frequency grid.

This ratio is denoted by $\tilde{h}(\delta,\varphi)$ and it is calculated as:

$$\tilde{h}(\delta,\varphi) = \\ = \frac{\pi}{T}\frac{|X_M(\hat{k}_0+1/2)| - |X_M(\hat{k}_0-1/2)|}{|X_M(\hat{k}_0+1/2)| + |X_M(\hat{k}_0-1/2)|}. \tag{11}$$

In order to study some important properties of this function, we will consider a noiseless signal and perform approximation by using the marginal-median DFT of continuous-time signals as:

$$h(\delta,\varphi) = \\ = \frac{\pi}{T}\frac{|X(\delta+\pi/T,\varphi)| - |X(\delta-\pi/T,\varphi)|}{|X(\delta+\pi/T,\varphi)| + |X(\delta-\pi/T,\varphi)|}. \tag{12}$$

For $(\delta,\varphi) \in [-\pi/T,\pi/T]\times[-\pi/2,\pi/2]$ it can be written as:

$$h(\delta,\varphi) = \\ \begin{cases} \frac{\pi}{T}\frac{1-\sqrt{2}\cos(|\delta|T/4+\pi/4)}{1+\sqrt{2}\cos(|\delta|T/4+\pi/4)}\mathrm{sign}(\delta) \\ \quad \text{for } |\delta| \in [\frac{\pi/T-4\min[|\varphi|,\pi/2-|\varphi|]}{T}, \frac{\pi}{T}] \\ \pi\frac{1+2\sin^2\varphi-\sqrt{(1+2\sin^2\varphi)^2-\sin^2\delta T/2}}{T\sin\delta T/2} \\ \quad \text{for } 0<|\delta| \le \frac{\pi}{T} - \frac{4|\varphi|}{T} \ \wedge\ |\varphi| \le \frac{\pi}{4} \\ \pi\frac{1+2\cos^2\varphi-\sqrt{(1+2\cos^2\varphi)^2-\sin^2\delta T/2}}{T\sin\delta T/2} \\ \quad \text{for } 0<|\delta| \le \frac{4|\varphi|}{T} - \frac{\pi}{T} \ \wedge\ |\varphi| \ge \frac{\pi}{4} \\ 0 \quad \text{for } \delta = 0. \end{cases} \tag{13}$$

Illustration of $h(\delta,\varphi)$ for $\varphi = 0$, $\varphi = \pi/8$ and $\varphi = \pi/4$ as function of δ (for $\delta = [0,\pi/T]$) is presented in Fig. 2a. Both $h(\delta,\varphi)$ and δ are normalized with T/π. The most important conclusion that can be drawn from this illustration is that $|h(\delta,\varphi)| \le |\delta|$ (i.e., lines representing $h(\delta,\varphi)$ are below the thin dotted line that represents $y=\delta$ in the entire interval and for all φ). Displacement can be calculated based on $h(\delta,\varphi)$ for a known φ as

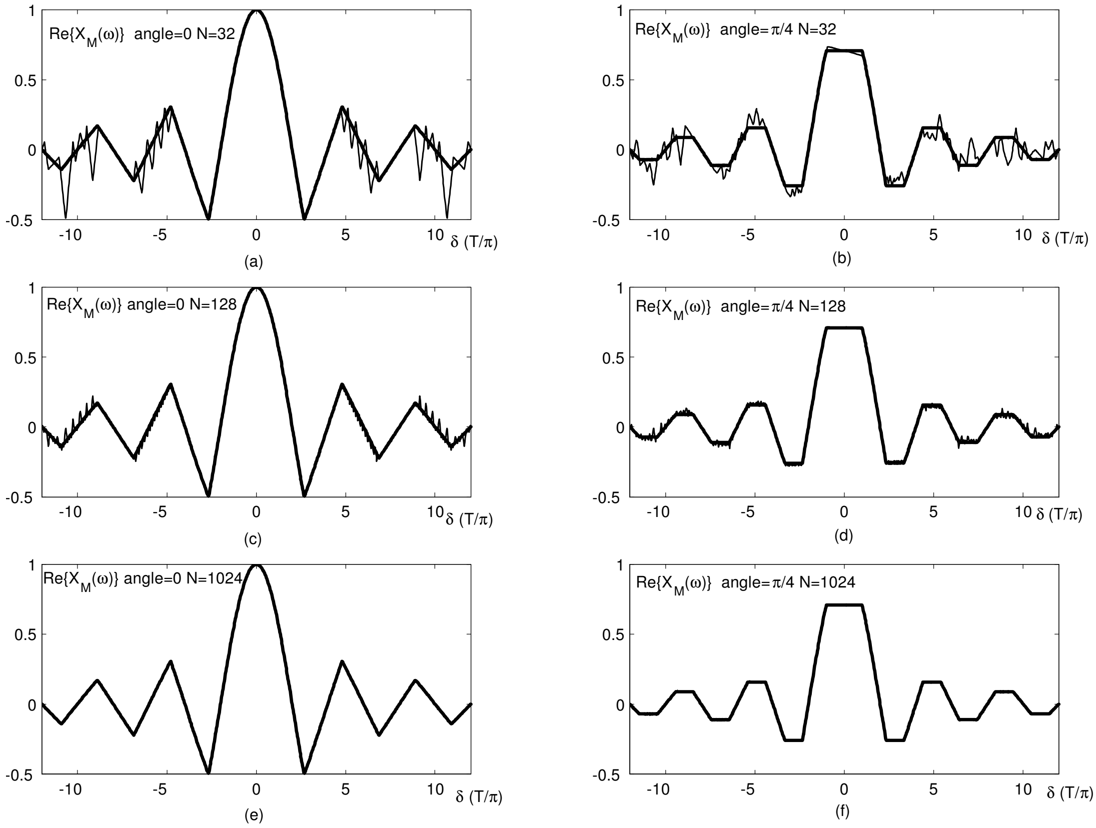

Fig. 1. Real part of marginal median DFT for various angles and number of samples. Thick line - derived expression for continuous-time median function; Thin lines - discrete time DFT.

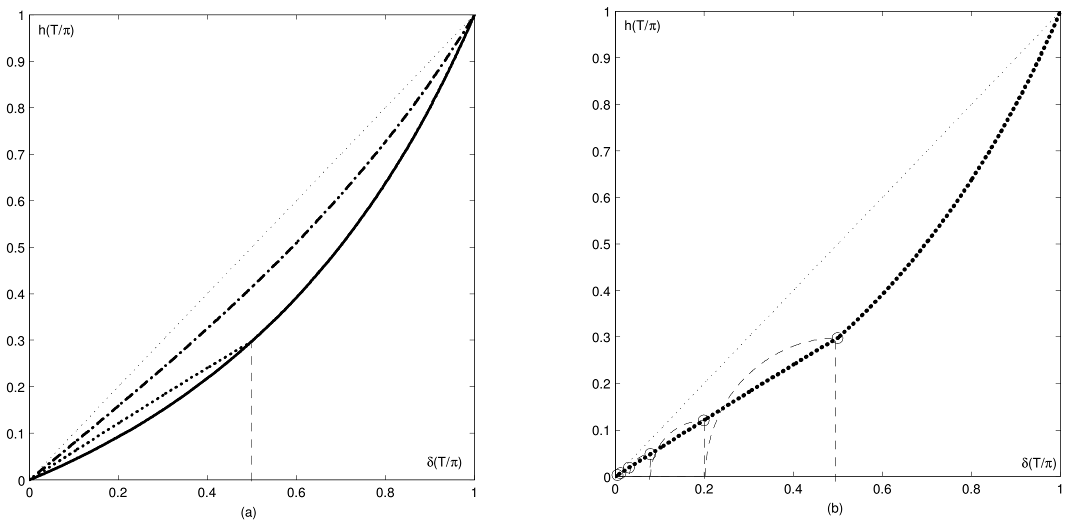

Fig. 2. (a) $h(\delta, \varphi)$ as function of δ for fixed φ. Thick solid line - $\varphi = \pi/4$, Thick dotted line - $\varphi = \pi/8$; Dash dotted line - $\varphi = 0$; Thin dotted line - function $y = \delta$. (b) Illustration of the iterative procedure for angle $\varphi = \pi/8$ and initial estimate displaced from the grid for $\delta = 0.5\pi/T$. Small circles represents iterations in the procedure.

$$\delta = \begin{cases} \left(\frac{4}{T}\arccos\frac{\sqrt{2}}{2}\frac{1-T|h(\delta,\varphi)|/\pi}{1+T|h(\delta,\varphi)|/\pi}-\pi/T\right) \\ \times\mathrm{sign}(h(\delta,\varphi)) \\ \quad \text{for } |h(\delta,\varphi)| \geq \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])}{1+\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])} \\ \frac{2}{T}\arcsin\frac{2\frac{T}{\pi}h(\delta,\varphi)(1+2\sin^2\varphi)}{(1+T^2h^2(\delta,\varphi)/\pi^2)} \\ \quad \text{for } |h(\delta,\varphi)| < \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])}{1+\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])} \\ \quad \wedge\ |\varphi| \leq \pi/4 \\ \frac{2}{T}\arcsin\frac{2\frac{T}{\pi}h(\delta,\varphi)(1+2\cos^2\varphi)}{(1+T^2h(\delta,\varphi)^2/\pi^2)} \\ \quad \text{for } |h(\delta,\varphi)| < \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])}{1+\sqrt{2}\sin(\min[|\varphi|,\pi/2-|\varphi|])} \\ \quad \wedge\ |\varphi| \geq \pi/4 \\ 0 \quad \text{for } h(\delta,\varphi)=0. \end{cases} \tag{14}$$

Since (11) is an approximation of $h(\delta,\varphi)$ and due to the noise presence we deal with the estimate $\hat{h}$ instead of the true value of $h(\delta,\varphi)$. Our goal is to perform precise estimation of frequency (i.e., displacement $\hat{\delta}$) from $\hat{h}$. Two algorithms are developed for this purpose in the next subsection. One of them is a direct application of (14) while the second is an iterative procedure inspired by [4].

B. Proposed techniques

B.1. Direct approach

The first approach uses directly expression (13) and estimates δ, based on the inverse formula (14). This procedure can be summarized as follows.

Step 1. Evaluation of the robust DFT in (6).

Step 2. Determination of the robust DFT maximum position (coarse frequency estimation):

$$\hat{k}_0 = \arg\max_k |X_M(k)|. \tag{15}$$

Step 3. Function $h(\delta,\varphi)$ is estimated by using two samples displaced from $\hat{k}_0$ for a half of the sampling interval:

$$\hat{h} = \frac{\pi}{T}\frac{|X_M(\hat{k}_0+1/2)|-|X_M(\hat{k}_0-1/2)|}{|X_M(\hat{k}_0+1/2)|+|X_M(\hat{k}_0-1/2)|}. \tag{16}$$

Step 4. Initial phase is estimated as:

$$\hat{\varphi} = \mathrm{phase}\left\{X_M(\hat{k}_0)\right\}. \tag{17}$$

Step 5. Displacement is estimated according to (14) by using $\hat{h}$ and $\hat{\varphi}$

$$\hat{\delta} = \begin{cases} \left(\frac{4}{T}\arccos\frac{\sqrt{2}}{2}\frac{1-T|\hat{h}|/\pi}{1+T|\hat{h}|/\pi}-\pi/T\right)\mathrm{sign}(\hat{h}) \\ \quad \text{for } |\hat{h}| \geq \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])}{1+\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])} \\ \frac{2}{T}\arcsin\frac{2\frac{T}{\pi}\hat{h}(1+2\sin^2\hat{\varphi})}{(1+T^2\hat{h}^2/\pi^2)} \\ \quad \text{for } |\hat{h}| < \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])}{1+\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])} \\ \quad \wedge\ |\hat{\varphi}| \leq \pi/4 \\ \frac{2}{T}\arcsin\frac{2\frac{T}{\pi}\hat{h}(1+2\cos^2\hat{\varphi})}{(1+T^2\hat{h}^2/\pi^2)} \\ \quad \text{for } |\hat{h}| < \frac{\pi}{T}\frac{1-\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])}{1+\sqrt{2}\sin(\min[|\hat{\varphi}|,\pi/2-|\hat{\varphi}|])} \\ \quad \wedge\ |\hat{\varphi}| \geq \pi/4 \\ 0 \quad \text{for } \hat{h}=0. \end{cases} \tag{18}$$

Step 6. Frequency is estimated as:

$$\hat{\omega} = \hat{k}_0\Delta\omega + \hat{\delta}. \tag{19}$$

B.2. Iterative approach

The second approach is related to the Aboutanios's iterative approach developed for the standard DFT. The single difference is in application of the marginal-median DFT, instead of the standard one.

Version of the iterative algorithm introduced by Aboutanios and applied to the robust DFT can be summarized as:

Step 1. Evaluation of the robust DFT in (6).

Step 2. Determination of the robust DFT maximum position (15).

Step 3. Set $\hat{\rho}_0 = 0$ and $i = 0$.

Step 4. Calculate

$$\hat{\rho}_{i+1} = \hat{\rho}_i - \frac{1}{\Delta\omega}h(\hat{\rho}_i) = \hat{\rho}_i - \frac{1}{2}\frac{|X_M(\hat{k}_0+\hat{\rho}_i+1/2)|-|X_M(\hat{k}_0+\hat{\rho}_i-1/2)|}{|X_M(\hat{k}_0+\hat{\rho}_i+1/2)|+|X_M(\hat{k}_0+\hat{\rho}_i-1/2)|}. \tag{20}$$

Step 5. Then set $i = i+1$ and repeat step 4.

Step 6. After a specific number of iterations, Q, the frequency of sinusoid is estimated as:

$$\hat{\omega} = \hat{k}_0\Delta\omega + \hat{\rho}_Q\Delta\omega. \tag{21}$$

Note that two different iterative approaches are proposed in [4], and we use the second one in (20). Intuitively, it can be concluded that this approach works accurately from the several reasons. The first point is that the robust DFT is accurate estimate of the standard DFT for non-noisy environment. Then it can be expected that the main effect in application of the robust DFT is in removing the impulse noise but not in changing other important properties that will affect accuracy of the iterative algorithm. The second, more important, point is that $h(\delta,\varphi)$ has the same sign and smaller magnitude than δ (see Fig. 2a). It means that any iteration $h(\hat{\rho}_i)$ will push δ closer toward true value but at the same time it will not exceed that value.

In order to show how the algorithm works, we illustrate non-noisy signal case with $\varphi = \pi/8$ in Fig. 2b. Assume that the result of coarse estimation is displaced from the true frequency for $\delta = 0.5[\pi/T]$. Then, the function $\hat{h}$ (i.e., $h(\hat{\rho}_0)$) is evaluated for this point (marked with a circle) producing the updated estimate. It can be seen that the algorithm converges toward the origin i.e., toward the true frequency. Accuracy of the algorithm and convergence will be considered in the next section.

Computational complexity of both proposed techniques depends on the marginal-median DFT. In the first technique, the marginal-median is calculated for $N+2$, while in the second approach it is evaluated for $N+2Q$ frequencies. The marginal-median means that we calculate two median functions (for real and imaginary parts) for sequences with N samples. Two strategies can be employed in this case for median evaluation:

- sorting of entire sequence by using some fast sorting procedure that typically requires $O(N\log_2 N)$ comparisons and selecting the median from them;
- applying the technique proposed in [12], with $O(N)$ comparisons for evaluation of median.

Since in our experiments $Q \ll N$, it can be concluded that the overall calculation complexity of both algorithms is $O(N^2)$ for median evaluated according to [12].

V. Numerical study

In our experiment we consider the signal:

$$f(t) = \exp(j\omega_0 t + j\delta t + j\varphi), \qquad (22)$$

within $t \in [-T/2, T/2)$ with $T = 2$. Experiments are performed with various numbers of samples in the interval $N \in \mathbf{N} = \{\mathbf{N_0 2^k}, \mathbf{k} = \mathbf{0}, \mathbf{1}, ..., \mathbf{6}\}$, where $N_0 = 32$. Here we demonstrate results achieved only with $N = 256$ and $N = 1024$. We selected $\omega_0 = 2\pi k_0/T$ where $k_0 = 12$. In each trial of our Monte Carlo simulations, δ and φ are selected randomly from the intervals $\delta \in [-\pi/T, \pi/T]$ and $\varphi \in [-\pi/2, \pi/2]$. Signal is embedded in the mixed Gaussian and impulse noise:

$$x(t) = f(t) + \nu_G(t) + \nu_I(t), \qquad (23)$$

where $\nu_G(t)$ is a white complex Gaussian noise with variance σ^2, while $\nu_I(t)$ is an impulse noise where impulses appear in both real and imaginary parts with probability p. We assume that negative and positive impulses appear with the same probability $p/2$, with the amplitude $\alpha = 5$. Real and imaginary parts of $\nu_I(t)$ are mutually independent[1].

The technique based on the standard DFT [4] is compared with two forms based on the marginal-median DFT: direct approach based on inverse function and iterative approach.

The mean squared error (MSE) for $N = 256$ and $N = 1024$ is depicted in Figs. 3 and 4. In both cases we consider the following environments: (a) Pure Gaussian noise as a function of SNR; (b) Mixed Gaussian and impulse noise for fixed amount of impulse noise $p = 5\%$, as a function of signal to Gaussian noise ratio; (c) Pure impulse noise as a function of p; (d) Mixed Gaussian and impulse noise for fixed amount of Gaussian noise $\sigma^2 = 0.25$ as a function of p. For pure Gaussian noise the iterative approach applied to the standard DFT outperforms other related techniques. However, it behaves worse than other techniques even for small amount of impulse noise (even for $p = 0.5\%$). In general, the iterative procedure applied on the marginal-median DFT

[1]In numerous papers it is assumed that disturbance can be considered to be of impulse nature if its magnitude is at least three signal amplitudes.

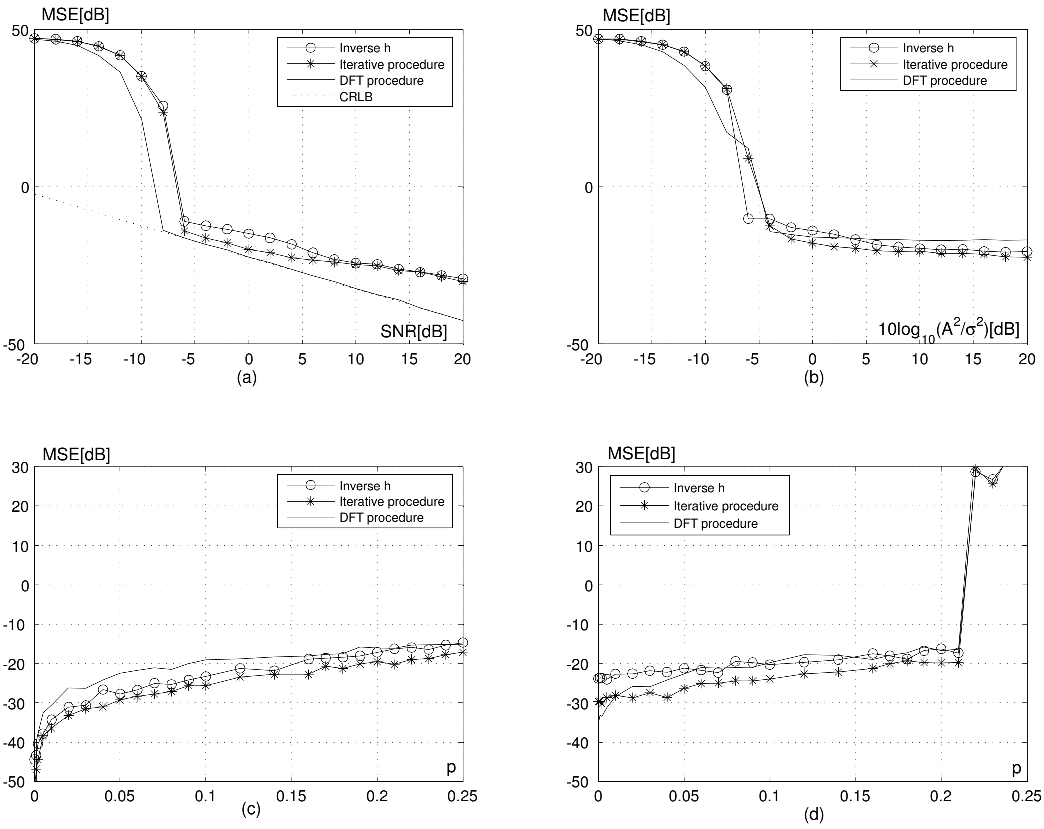

Fig. 3. Mean squared error in frequency estimation for $N = 256$ samples: Solid line - iterative procedure for standard DFT; Solid line and * sign - iterative procedure for robust DFT; Solid line and o sign - inverse $h(\delta, \varphi)$ evaluation; Dotted line - CRLB for Gaussian noise. (a) Pure Gaussian noise as function of SNR; (b) Mixed Gaussian and impulse noise for fixed percentage of impulses 5% as function of ratio of signal power and variance of Gaussian noise; (c) Pure impulse noise as function of percentage of noise; (d) Mixed Gaussian and impulse noise for fixed variance of the Gaussian noise $\sigma^2 = 0.25$ as function of percentage of impulse noise.

behaves better than the inverse evaluation of $h(\delta, \varphi)$.

For example, for mixed Gaussian and impulse noise with ratio between signal amplitude and Gaussian noise of 7dB and for $p = 5\%$, the iterative procedure applied on the robust DFT produces better results than the evaluation of inverse function $h(\delta, \varphi)$ (for 1.5dB for $N = 256$ and 2.8dB for $N = 1024$) or the iterative procedure applied on the standard DFT (for 3.5dB for $N = 256$ and for 3.2dB for $N = 1024$). For pure impulse noise with $p = 10\%$ of impulses improvement is even larger: 2dB for $N = 256$ and $N = 1024$ with respect to the inverse function, 7dB for $N = 256$ and for 6.4dB for $N = 1024$ with respect to the iterative procedure applied on the standard DFT.

In addition, experiments related to dependence of the MSE of δ are performed. The MSE as a function on δ has been depicted in Fig. 3 for $N = 256$. Results are obtained by Monte Carlo simulations with 1000 trials for each considered δ. Two noise environments are considered: Gaussian noise environment with signal to noise ratio SNR=0dB (Fig. 5a) and pure impulse noise environment with 2% of impulses in both real and imaginary parts of signal (Fig. 5b). It can be seen from Fig. 5a that the iterative algorithm applied to the

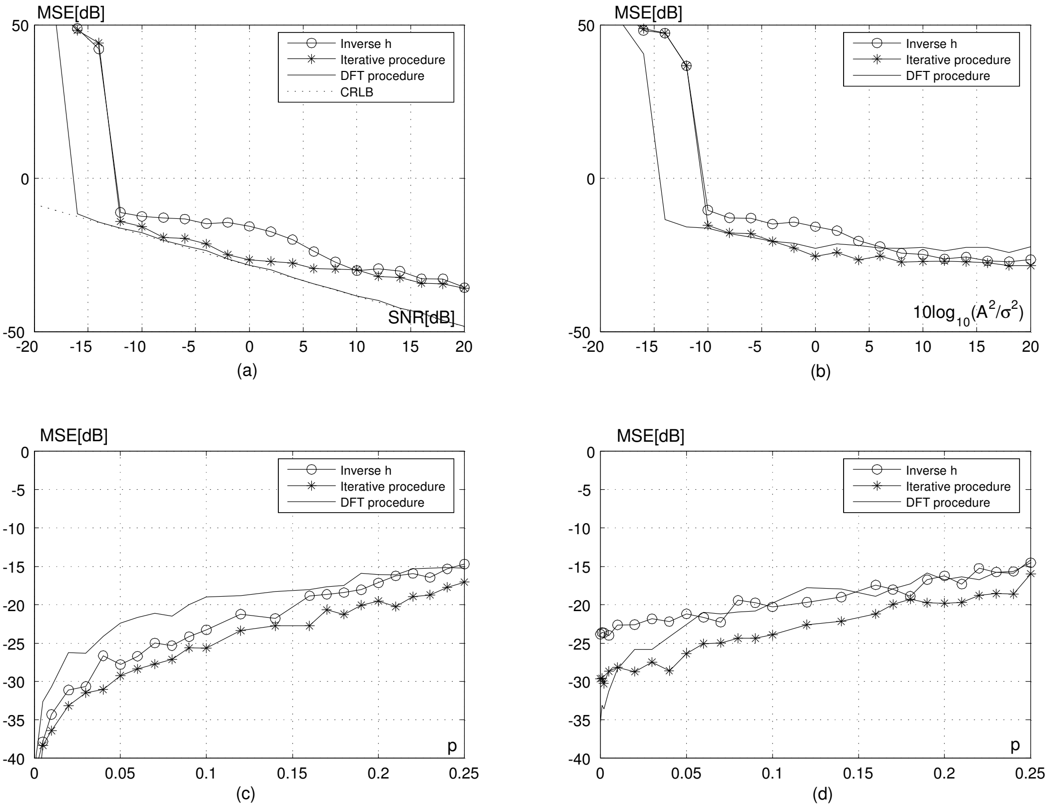

Fig. 4. Mean squared error in frequency estimation for $N = 1024$ samples: Solid line - iterative procedure for standard DFT; Solid line and * sign - iterative procedure for robust DFT; Solid line and o sign - inverse $h(\delta, \varphi)$ evaluation; Dotted line - CRLB for Gaussian noise. (a) Pure Gaussian noise as function of SNR; (b) Mixed Gaussian and impulse noise for fixed percentage of impulses 5% as function of ratio of signal power and variance of Gaussian noise; (c) Pure impulse noise as function of percentage of noise; (d) Mixed Gaussian and impulse noise for fixed variance of the Gaussian noise $\sigma^2 = 0.25$ as function of percentage of impulse noise.

robust DFT converges for Gaussian noise environment for just two iterations. Namely, for $Q = 2$ we obtain results of the same order of accuracy as for $Q > 2$. Also, it can be seen that this approach is significantly better than estimation of displacement based on inverse formula (18). An important fact is that the MSE for $Q = 2$ is approximately constant for $\delta \in [-\pi/T, \pi/T]$. However, direct evaluation of the inverse formula produces highest error for $\delta \approx \pm 0.3\pi/T$. It resulted in better accuracy of the iterative procedure by 5dB comparing to the inverse evaluation for this environment. Note that the iterative procedure applied on the standard DFT for Gaussian noise environment requires $Q = 2$ iterations to converge toward accurate results, i.e., the same as in the case of the robust DFT.

In the impulse environment at least $Q = 5$ iterations is required for the iterative procedure to converge. Here, direct application of (18) produces excellent results very close to those of the iterative procedure after $Q = 5$ iterations (only by 1.5dB worse). Results of this experiment confirm that convergence of the iterative procedure applied to the robust DFT for impulse noise environment is slower than in the case of the standard DFT for signals corrupted by Gaussian noise. Then, simpler procedure related to inverse evaluation of $h(\delta, \varphi)$ could be used in the case when computation time is critical.

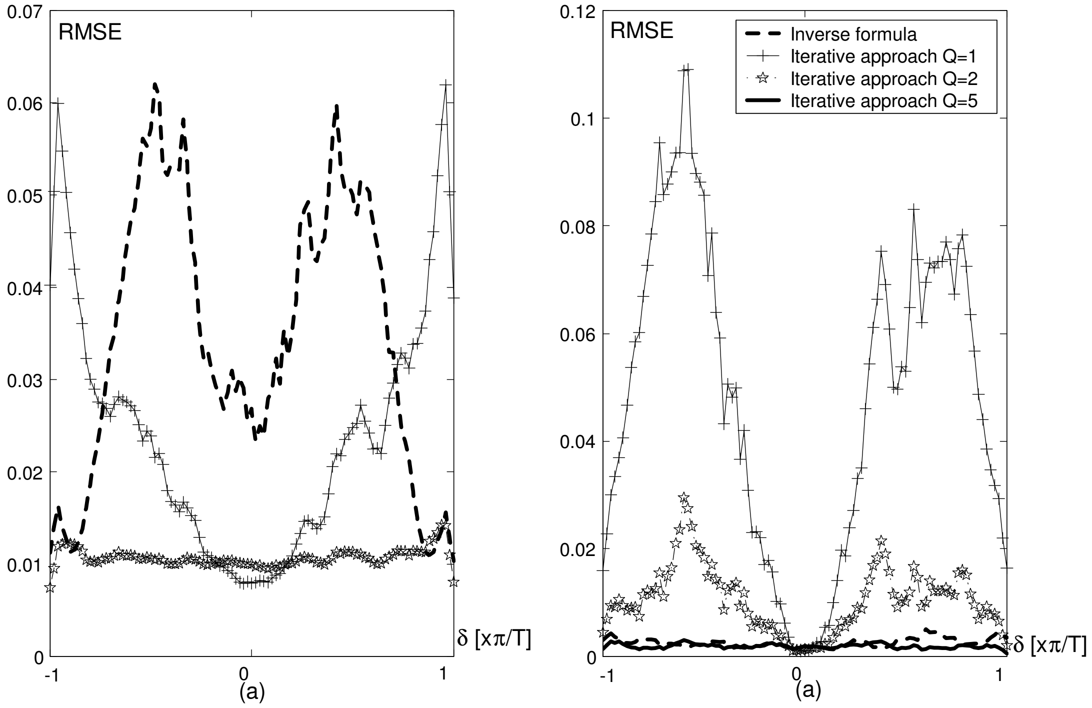

Fig. 5. MSE as function of δ for $N = 256$ for two noise environments: (a) Gaussian noise SNR=0dB; (b) Impulse noise environment with 2% of impulses in real and imaginary parts.

VI. Conclusion

Two procedures for precise estimation of sinusoidal signals parameters for signals embedded in impulse noise environments are proposed. These procedures have two stages: coarse stage where estimation is performed by using position of the robust DFT maximum and a fine stage where specific ratio of robust DFT samples close to maximum obtained in coarse stage is used. The first procedure is based on evaluation of the inverse function $\hat{\delta} = h^{-1}(\delta, \varphi)$ for estimated $\hat{\varphi}$, while the second approach is an iterative procedure. Better results are achieved with the iterative procedure. In our experiments, it converges for 2-7 iterations. Accuracy improvement with respect to the iterative procedure applied on the standard DFT is about 6dB for signals corrupted with more than 5% of impulses. In future research we will consider precise estimation of signal parameters for other robust DFT forms (especially for L-filter form).

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