Modifications of the Cubic Phase Function

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Abstract— **By introducing a symmetric pair of time instants, a modification of the cubic phase (CP) function, named as the quartic phase (QP) function, is proposed to estimate the quadratic FM signal. The performance in terms of estimate bias and variance is presented via the first-order permutation principle. Two extensions are presented for multiple components and the cubic FM signals. Both theoretical analysis and numerical examples confirm that the QP function and its extensions provide a number of advantages, such as
1. lower asymptotic mean-square error (MSE) for the estimate of the third-order phase parameter at high SNR;
2. a better capability of discriminating multi-component signals;
3. a lower SNR threshold for the estimates of the cubic FM signal.**

I. INTRODUCTION

The frequency-modulated (FM) signal modeling can be found in a number of applications such as radar, communications, music, speech, geophysics, and biomedicine [1]-[5]. In these applications, signals having the polynomial phase with low order, i.e., the linear, quadratic and cubic FM signals (corresponding to the 2nd-, 3rd- and 4th-order polynomial phase signal (PPS)), are the most frequently encountered. In the literature, the case of the linear FM signal has been thoroughly studied [6]-[8], however, parameter estimation of the quadratic and cubic FM signals is still a challenge.

The most accurate way for analyzing the quadratic FM signal is the maximum likelihood (ML) estimation [5]. It yields optimal results but requires a three-dimensional maximization, and thus it is computationally exhausting. To avoid the multidimensional search, a number of suboptimal approaches were proposed, for example, the high-order

ambiguity function (HAF) [3], [9], the integrated general ambiguity function (IGAF) [10] and the product HAF (PHAF) [11]. The main idea behind the HAF-based method is to iteratively transform the signal to obtain a sinusoid at a certain frequency related to the phase parameters. Meanwhile, the polynomial Wigner-Ville distribution (PWVD) was proposed for the high-order FM signal, i.e., the quadratic and cubic FM signal [12], [13]. The kernel of the PWVD ensures a time-varying sinusoid at the frequency related to the instantaneous frequency (IF). Recently, a bilinear transform - the cubic phase (CP) function was proposed by introducing the instantaneous frequency rate (IFR) in [4] and [14]. For a quadratic FM signal defined as

$$s(t) = Ae^{j\phi(t)} = Ae^{j(a_0 + a_1 t + a_2 t^2 + a_3 t^3)}, \quad (1)$$

$$-T/2 \le t \le T/2$$

where A, $\phi(t)$, and $\{a_i\}_{i=0}^3$ are the amplitude, phase, and phase coefficients, respectively, the CP function is presented as

$$CP(t, \Omega) = \int_{\tau=0}^{T/2} s(t+\tau)s(t-\tau)e^{-j\Omega\tau^2}d\tau. \quad (2)$$

Substituting (1) for $s(t)$ in (2), the resulting signal of the bilinear transform is

$$s(t + \tau)s(t - \tau) = A^2 e^{j2[\phi(t) + (a_2 + 3a_3 t)\tau^2]}. \quad (3)$$

From (2) and (3), the CP function will have a peak at $2(a_2 + 3a_3 t)$, which is the IFR of the signal in (1). Once the IFR is obtained, the phase parameters, a_2 and a_3, can be estimated by selecting two time positions and solving the resulting equations. In this paper, we present a modification of the CP function by using two time instants which are symmetric with respect to zero. This modification results in

the quartic phase (QP) function. The theoretical analysis shows that, with respect to the Cramér-Rao lower bounds (CRLB), the QP function has a closer asymptotic mean-square error (MSE) than other existing methods at high SNR. Two extensions are introduced to discriminate multicomponent signals and estimate the parameters of the cubic FM signal.

This paper is organized as follows. Section II presents the QP function. In section III, the asymptotic statistical result for the a_3 estimate is derived. Extensions for the case of multiple components and the case of the cubic FM signal are described in Section IV. Section V provides some numerical examples to evaluate the performance of the QP function and corresponding extensions. Finally, conclusions are drawn in Section VI.

II. The Proposed Algorithm

By introducing a symmetric pair of time instants, the QP function is defined as

$$\text{QP}(\text{t}, \omega) = \int_{\tau=0}^{\text{T}/2} \text{s}(\text{t}+\tau)\text{s}(\text{t}-\tau)$$

$$\times s^*(-t+\tau)s^*(-t-\tau)e^{-j\omega\tau^2} d\tau. \quad (4)$$

where t is assumed to be positive.
From (2) and (4), the significant difference is the employing nonlinear transform. The CP function employs a bilinear transform and the QP function involves in a fourth-order nonlinearity. In the following, we will introduce the QP function in two steps: the nonlinear transform and the quadratic phase filter.

A. The Fourth-order Nonlinear Transform

For an arbitrary signal with phase $\phi(t)$, assume that $\phi_1 = \text{phase}[\text{s}(\text{t}+\tau)]$, $\phi_2 = \text{phase}[\text{s}(\text{t}-\tau)]$, $\phi_3 = \text{phase}[\text{s}(-\text{t}+\tau)]$, $\phi_4 = \text{phase}[\text{s}(-\text{t}-\tau)]$, and $\phi_{\text{QP}} = \text{phase}[\text{s}(\text{t}+\tau)\text{s}(\text{t}-\tau)s^*(-\text{t}+\tau)s^*(-\text{t}-\tau)]$, where phase[.] denotes the phase extractor. Using the Taylor series expansion and setting the expansion order as M, we obtain

$$\phi_1 + \phi_2 = \sum_{l=0}^{M/2} \frac{2\phi^{(2l)}(t)\tau^{2l}}{(2l)!}; \quad (5)$$

$$\phi_3 + \phi_4 = \sum_{l=0}^{M/2} \frac{2\phi^{(2l)}(-t)\tau^{2l}}{(2l)!}; \quad (6)$$

$$\phi_{\text{QP}} = (\phi_1 + \phi_2) - (\phi_3 + \phi_4) =$$

$$\sum_{l=0}^{M/2} \frac{2[\phi^{(2l)}(t) - \phi^{(2l)}(-t)]\tau^{2l}}{(2l)!}. \quad (7)$$

Substituting $\sum_{i=0}^{P} a_i t^i$ for $\phi(t)$, where P is the phase order and letting $\eta(\phi) = \phi^{(2l)}(t) - \phi^{(2l)}(-t)$ yield

$$\eta(\phi) = \begin{cases} \sum_{v=l}^{P/2-1} \frac{2a_{2v+1}t^{2v-2l+1}(2v+1)!}{(2v-2l+1)!} & P \text{ is even;} \\ \sum_{v=l}^{(P-1)/2} \frac{2a_{2v+1}t^{2v-2l+1}(2v+1)!}{(2v-2l+1)!} & P \text{ is odd.} \end{cases} \quad (8)$$

Using (8), (7) can be expressed as

$$\phi_{\text{QP}} = \begin{cases} \sum_{l=0}^{P/2}\sum_{v=l}^{P/2-1} \frac{4a_{2v+1}t^{2v-2l+1}\tau^{2l}(2v+1)!}{(2l)!(2v-2l+1)!} & P \text{ is even;} \\ \sum_{l=0}^{\frac{P-1}{2}}\sum_{v=l}^{\frac{P-1}{2}} \frac{4a_{2v+1}t^{2v-2l+1}\tau^{2l}(2v+1)!}{(2l)!(2v-2l+1)!} & P \text{ is odd.} \end{cases} \quad (9)$$

For a quadratic FM signal, (9) reduces to

$$\phi_{\text{QP}} = 4(a_1 t + a_3 t^3) + 12a_3 t\tau^2. \quad (10)$$

It can be said that the multilinear transform converts the quadratic FM signals into a space that, at any given value of time sets, has a quadratic term in τ and another invariant to τ. In particular, the quadratic phase coefficient of the resulting signal is $12a_3 t$. With the knowledge on this coefficient, we can estimate the parameter a_3.

B. The Quadratic Phase Filter

In order to obtain the quadratic phase coefficient, a quadratic phase filter is applied to compensate the quadratic phase term in τ [14]. Using the identity [15]

$$\int_{-\infty}^{+\infty} e^{-j\tau t^2} dt = \sqrt{\frac{\pi}{\tau}}e^{-j(\pi/4)}, \quad \tau > 0, \quad (11)$$

we obtain

$$|\text{QP}(\text{t}, \omega)| = \frac{\text{A}^4}{2}\sqrt{\frac{\pi}{|12\text{a}_3\text{t} - \omega|}}. \quad (12)$$

It can be concluded that the QP function maximizes along $\omega = 12a_3t$, while dispersing for other ω. The parameter a_3 can be estimated once a distinct peak is detected. Using the nonlinear least squares, the estimate of a_3 is given as

$$\hat{a}_3 = \frac{\arg\max_\omega |QP(t, \omega)|}{12t}. \qquad (13)$$

Generally, the time instant, t, determines the variance of the a_3 estimate. Based on the statistical analysis (see Appendix for the details), if the time set is chosen to be $n \approx 0.2291N$ in discrete time (N is the number of samples), the a_3 estimate achieves the minimum asymptotic mean-square error at high SNR [19].

C. Implementation

The implementation of the QP-based method is shown as follows. The first step is to determine the a_3 estimate at a time position by extracting the peak from the QP function. Once a_3 has been obtained, the observation can be appropriately dechirped to a chirp signal and a conventional techniques can then be used to estimate the parameters of the resulting chirp signal.

Directly computing (4) requires about $O(N^2)$ operations. Motivated by the fast implementation of the CP function, maximization of the QP function can be reduced to $O(N \log_2 N)$ operations using the subband decomposition techniques [4].

III. Statistical Analysis of the a_3 Estimates

Since the estimation algorithm is iterative, it inevitably suffers from the error propagation effect. That is error in the a_3 estimate will propagate to the latter estimates. Hence, the statistical analysis of the a_3 estimate is the most crucial part in this paper, while other estimates can be analyzed in a similar way in [9] and [16].

There are three existing methods for parameter estimation of the quadratic FM signal, i.e., the HAF, PWVD and CP function. Table I lists the asymptotic MSE of these methods for the a_3 estimate at high SNR. The HAF-based method maybe the most frequently used. However, the inherent eighth-order nonlinearity in the HAF for the quadratic FM signal results in high asymptotic MSE. Quantificationally, the HAF-based asymptotic MSE is about 43.17% higher than the QP function with respect to the CRLB. Moreover, the high-order nonlinearity in the HAF gives rise to high SNR threshold (see Section V for the details). In the literature of time-frequency analysis, the PWVD is adaptive to high-order FM signal the quadratic time-frequency distribution such as the Wigner-Ville distribution. The QP function outperforms the PWVD in terms of the SNR threshold and asymptotic MSE, due to the fact that there is sixth-order nonlinearity in the PWVD [4]. The most competitive method to the QP function is the standard CP function, since the CP function involves in only a second-order nonlinearity, which leads to lower SNR threshold than the QP function. However, with respect to the CRLB, the asymptotic MSE of the a_3 estimate using the QP function is about 32.53% lower than the CP function at high SNR.

IV. Algorithm Extensions

The above content established the QP-based method for the monocomponent quadratic FM signal. In the following, the QP-based method is simply modified for the case of multicomponent signals and the case of the cubic FM signal. Specifically, we note that the cubic FM signal has two practical applications, which are described in [17].

A. Multicomponent Case

It has been shown in [18] that for multicomponent signals the distinct cross-terms or spurious peaks occur producing problem with identification of parameters of signal components using the CP function. The QP function with simply modification can be extended for multicomponent case. To discern the auto-terms from the cross-terms or possible spurious peaks, it is better to make use of the time dependence. From (12), it is clear that the auto-terms are linearly related to the time position, i.e. $\omega = 12a_3n$. However, the cross-terms have not this type of time dependence, i.e., nonlinear to the time. By using the spec-

TABLE I
THEORETICAL MSE FOR THE a_3 ESTIMATE AT HIGH SNR

The Estimate	QPF	CPF	HAF	CRLB
a_3	$\frac{1582.5}{N^7 SNR}$	$\frac{2038}{N^7 SNR}$	$\frac{2187}{N^7 SNR}$	$\frac{1400}{N^7 SNR}$
P_s	13.04%	45.57%	56.21%	-

tral scaling technique introduced in the PHAF [11], the product form of QP functions is defined as

$$\mathrm{PQP}(\omega; n_L) = \prod_{l=1}^{L} \mathrm{QP}(n_l, \frac{n_L}{n_l}\omega). \qquad (14)$$

It can be said from (14) that the spectral scaling operation ensures the peaks are properly aligned at $\omega = 12a_3 n_L$. When the auto-terms are aligned, the subsequent multiplication amplifies the auto-terms and weakens the cross-terms that are misaligned. In order to simplify the implementation of the PQP function, the set of time instants can be selected as $n_L = 2n_{L-1} = 4n_{L-2} = \cdots = 2^{L-1}n_1$, followed by an interpolation with order $2, 4 \cdots, 2^{L-1}$.

B. Parameter Estimation of the Cubic FM signal

Conventional techniques such as the HAF for the cubic FM signal first estimate the highest-order phase coefficient, i.e., a_4, dechirp the observations with the estimate, and repeat the above procedure until the spectrum does not present non-zero peak. In contrast to the conventional techniques, the QP-based method makes use of the QP function to extract and estimate the a_3 other than a_4 with (10) which holds for the cubic FM signal as well. Once a_3 is obtained, the dechirp technique is used and the resulting signal can be approximated as $s_d(t) = Ae^{j(a_0 + a_1 t + a_2 t^2 + a_4 t^4)}$. To estimate the a_4 from the dechirped signal $s_d(t)$, we apply a modified QP function with two time instants one of which is zero. The modified QP function can be defined as

$$\mathrm{QP}_m(t, \omega) =$$

- HAF: $a_4 \to a_3 \to a_2 \to a_1 \to a_0 \to A$
  8th-order nonlinearity
- HPF: $a_4, a_3, a_2 \to a_1 \to a_0 \to A$
  6th-order nonlinearity
- QPF: $a_3 \to a_4 \to a_2 \to a_1 \to a_0 \to A$
  4th-order nonlinearity

Fig. 1. The estimate procedure for the cubic FM signal

$$= \int_{\tau=0}^{T/2} s_d(t+\tau) s_d(t-\tau) s_d^*(0+\tau) s_d^*(0-\tau) e^{-j\omega\tau^2} d\tau. \qquad (15)$$

Taking the dechirped signal into (15) yields

$$|\mathrm{QP}_m(t, \omega)| = \frac{A^4}{2}\sqrt{\frac{\pi}{|12a_4 t^2 - \omega|}}. \qquad (16)$$

Hence, the value ω that maximizes the modified QP function in (16) can determine the a_4.

Compared with other techniques for the cubic FM signal, the QP-based method involves in only a fourth-order nonlinearity that is lower than a sixth-order nonlinearity in the higher-order phase function (HPF: higher-order version of the CP function) [4] and the PWVD [12], and an eighth-order nonlinearity in the HAF [3]. As a consequence, the QP-based method allows parameter estimation at low SNR. The procedures of the above methods for the cubic FM signal are compared in Fig. 1.

V. SIMULATIONS

Example 1: In order to directly compare with other methods, the tested signal in this example is the same quadratic FM used in [3, Sect. IV.A] and [4, Sect. IV]. The SNR is incremented in 1 dB interval from -5 to 20 dB, the sampling interval is 1, and the number of samples is $N = 257$. The signal parameters are $A = 1, a_3 = \pi 10^{-5}, a_2 = -\pi 10^{-3}, a_1 = 0.3\pi$, and $a_0 = 0$. At each SNR, 200 runs of the Monte Carlo simulation are performed. The
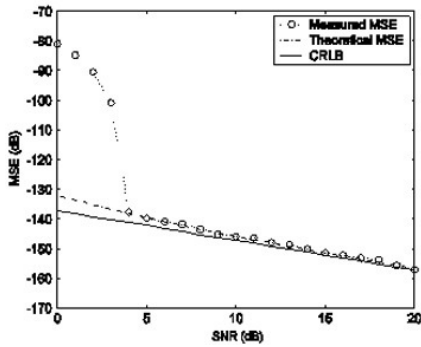
Fig. 2. Comparisons between theoretical and measured MSE for the a_3 estimate of the quadratic FM signal

MSEs of the a_3 estimate are plotted in Fig. 2. The measured MSEs are indicated with circles, whereas corresponding asymptotic MSEs are shown as dotted lines. The straight line in this plot is the CRLB for the a_3 estimate. It confirms that the simulation results adhere to the theoretical analysis above the SNR threshold which is about 4 dB.

Fig. 2 presents the performance comparisons among the above three methods. In this plot, we define a term P_s indicating how far from the CRLB to the asymptotic MSE:

$$P_s = \frac{\text{AsymptoticMSE}}{\text{CRLB}} - 1. \qquad (17)$$

Obviously, larger P_s means corresponding MSE is further from the CRLB. The theoretical and measured P_s for the HAF, CP function and QP functions above the SNR threshold are shown in Fig. 3. At high SNR, the P_s is also listed in Table I. It verifies that the measured MSE for the QP function is generally lower than that of the CP function above 4 dB and the HAF at all SNR. Note that the fluctuation of the measured MSE can be observed in Fig. 3. This is because the term P_s magnifies the errors between the theoretical and measure MSEs.

Example 2: This example presents the PQP function applied to multicomponent 1) quadratic FM signals and 2) cubic FM signals. The parameters of two quadratic FM signals are
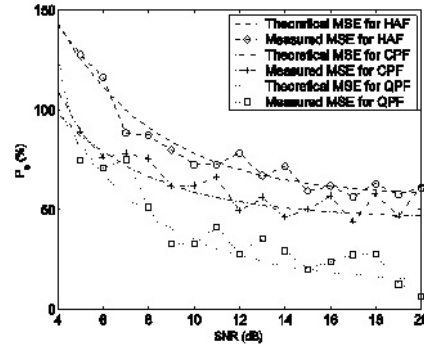- 1st component: $A_1 = 1, a_{10} = 0, a_{11} = \pi/5, a_{12} = -2\pi/(5N), a_{13} = \pi/(5N^2)$;

Fig. 3. Performance comparisons among the HAF, the CP function and QP function for the a_3 estimate above 3 dB
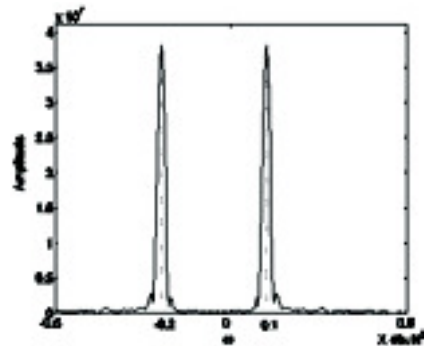

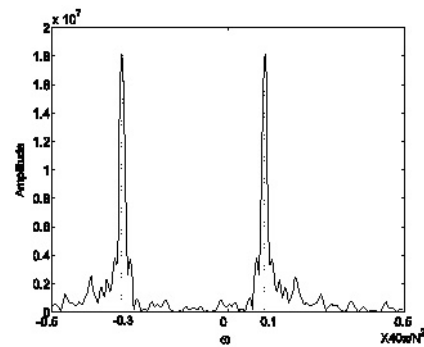Fig. 4. Product QP function with $L = 3$ for two-component quadratic FM signals

Fig. 5. Product QP function with $L = 3$ for two-component cubic FM signals
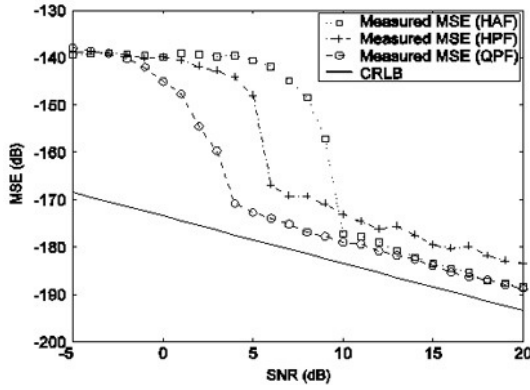
Fig. 6. Performance comparisons among the HAF, HP function and QP function for the a_4 estimate of the cubic FM signal

• 2nd component: $A_2 = 1, a_{20} = 0, a_{21} = -2\pi/5, a_{22} = \pi/(5N), a_{23} = -2\pi/(5N^2)$.

The set of time instants in the PQP function is [5, 10, 20], and $L = 3$. The result is shown in Fig. 4. In this plot, two distinct peaks can easily be observed at $\omega_1 = \pi/(5N^2) \times 20$ and $\omega_2 = -2\pi/(5N^2) \times 20$. Then we apply the PQP function to two-component cubic FM signals with parameters:

• 1st component: $A_1 = 1, a_{10} = 0, a_{11} = \pi/5, a_{12} = -2\pi/(5N), a_{13} = \pi/(5N^2), a_{14} = -\pi/(5N^3)$;

• 2nd component : $A_2 = 1, a_{20} = 0, a_{21} = -2\pi/5, a_{22} = \pi/(5N), a_{23} = -3\pi/(5N^2), a_{24} = 3\pi/(5N^3)$.

The set of time instants is [5, 10, 20], and $L = 3$. Once again, two distinct peaks can be observed in Fig. 5. The spectral positions corresponding to two peaks are $\omega_1 = \pi/(5N^2) \times 20$ and $\omega_2 = -3\pi/(5N^2) \times 20$.

Example 3: To evaluate the a_4 estimate for the cubic FM signal, 200 runs of the Monte Carlo simulation are performed. The selected signal is the first cubic FM signal in Example 2. The results are shown in Fig. 6. In this plot, the lowest threshold SNR around 4dB is derived by using the QP-based method. It is about 2dB and 6dB lower than the SNR threshold of the HPF and HAF. Moreover, when the SNR is higher than the threshold, the measured MSE for the QP-based method is generally lower than other methods.

VI. Conclusion

A modification of the CP function has been proposed for parameter estimation of a quadratic FM signal. It utilizes a symmetric pair of time instants and employs a fourth-order nonlinear transform. Statistical analysis shows that the variance of the a_3 estimate is only 13.04% higher than the CRLB at high SNR. The extensions for multicomponent case and parameter estimation of the cubic FM signal are also discussed. The simulation results adhere to the theoretical analysis.

Appendix

This appendix provides the first-order permutation analysis for the a_3 estimate. For brevity, we use the same notation of symbols in [4] (see Appendix I of [4]).

In order to apply the general formulae in Appendix I of [4], it is necessary to reassign the variables and equations corresponding to the QP-based method:

$$g_N(\omega) = \text{QP}_s(n, \omega), \qquad (18)$$

where the QP_s represents the QP function of the noiseless signal $s(n)$.

The perturbation to $g_N(\omega)$ provided by addition of noise, $v(n)$, to $s(n)$ is

$$\delta g_N(\omega) = \sum_{m=0}^{(N-1)/2-n} z_{vs}(n, m) e^{-j\omega m^2}, \quad (19)$$

where $z_{vs}(n, m)$ approximates the interference terms containing not more than two noise factors. Note that n is assumed to be positive.

The function $g_N(\omega), \delta g_N(\omega)$, and their derivatives, evaluated at the point of global maximum $\omega_0 = 12a_3n$, are given by

$$g_N(\omega_0) \approx A^4 K(N/2 - n), \qquad (20)$$

$$\frac{\partial g_N(\omega_0)}{\partial \omega} \approx -jA^4 K \frac{(N/2 - n)^3}{3}, \qquad (21)$$

$$\frac{\partial^2 g_N(\omega_0)}{\partial \omega^2} \approx -A^4 K \frac{(N/2 - n)^5}{5}, \qquad (22)$$

$$\delta g_N(\omega_0) \approx \sum_{m=0}^{N/2-n} z_{vs}(n, m) e^{-j\omega_0 m^2}, \qquad (23)$$

$$\frac{\partial \delta g_N(\omega_0)}{\partial \omega} \approx -j \sum_{m=0}^{N/2-n} m^2 z_{vs}(n,m) e^{-j\omega_0 m^2},$$
(24)

where $K = e^{j4[a_1 n + a_3 n^3]}$.

Substituting the above results for corresponding symbols of Appendix I in [4] yields

$$\alpha \approx \frac{-8A^8(N/2-n)^6}{45},$$
(25)

$$\beta \approx -2A^4(N/2-n)[\Im\{\Gamma\}],$$
(26)

where

$$\Gamma \approx K \sum_{m=0}^{N/2-n} \left(m^2 - \frac{(N/2-n)^2}{3} \right) z_{vs}^*(n,m) e^{j\omega_0 m^2},$$
(27)

and $\Im(\cdot)$ denotes the imaginary part of $(\cdot)$. Henceforth, we have

$$\delta\omega \approx -\frac{45 \cdot \Im\{\Gamma\}}{4A^4(N/2-n)^5}.$$
(28)

Its expectation, the bias of the a_3 estimate, is asymptotical zero (to the first-order approximation).

For the mean-square of (28), it is necessary to compute the values of $E\{\Gamma\Gamma^*\}$ and $E\{\Gamma\Gamma\}$. With some tedious but straight computations, we get

$$E\{\Gamma\Gamma^*\} \approx \frac{8}{45}(2A^6\sigma^2 + 3A^4\sigma^4)(N/2-n)^5,$$
(29)

$$E\{\Gamma\Gamma\} \approx \frac{1}{180}\left(2A^6\sigma^2 + A^4\sigma^4\right)\varphi u\,(N-4n),$$
(30)

where $\varphi = N^5 - 20nN^4 + 120n^2N^3 - 240n^3N^2 + 80n^4N - 64n^5$, and $u(\cdot)$ denotes the unit step function. Subsequently,

$$E\left\{(\delta\omega)^2\right\} \approx \frac{E\{\beta^2\}}{\alpha^2}$$
(31)

where $\text{SNR} = A^2/\sigma^2$.

Taking into account $\delta a_3 = \delta\omega/12n$, we get

$$E\{(\delta a_3)^2\} \approx \frac{5}{1024n^2(N/2-n)^5\text{SNR}}$$

$$\times \left[16\left(2 + \frac{3}{\text{SNR}}\right) \right.$$

$$\left. - \left(1 + \frac{1}{2\text{SNR}}\right)\frac{\varphi}{(N/2-n)^5}u\,(N-4n) \right].$$
(32)

It is shown that the variance of the a_3 estimate depends on the values of N, SNR and n. For any given N and SNR, $E\left\{(\delta a_3)^2\right\}$ can be minimized by choosing n. Numerical study shows that $n \approx 0.2291N$ gives the minimum variance of the a_3 estimate at high SNR. Therefore, the recommended choice of time instant is $n = 0.2291N$. When $n = 0.2291N$, the minimum asymptotic MSE for the a_3 estimate is

$$E\{(\delta a_3)^2\} \approx \frac{1582.5 + \frac{2829.7}{\text{SNR}}}{N^7\text{SNR}}.$$
(33)

REFERENCES

[1] A. W. Rihaczek, "Principles of High-Resolution Radar," CA: Peninsula, 1985.

[2] J. C. Curlander and R. N. McDonough, "Synthetic Aperture Radar - System and Signal Processing," John Wiley & Sons, New York, 1991.

[3] S. Peleg and B. Friedlander, "The discrete polynomial phase transform," IEEE Trans. Signal Processing, Vol. 43, No. 8, pp. 1901-1914, 1995.

[4] P. O'Shea, "A fast algorithm for estimating the parameters of a quadratic FM signal," IEEE Trans. Signal Processing, Vol. 52, No. 2, pp. 385-393, 2004.

[5] S. Barbarossa, and V. Petrone, "Analysis of polynomial phase signals by an integrated generalized ambiguity function," IEEE Trans. Signal Processing, Vol. 47, No. 2, pp. 316-327, 1997.

[6] T. Abotzoglou, "Fast maximum likelihood joint estimation of frequency and frequency rate," IEEE Tran. Acoust., Speech, Signal Processing, Vol. AES-22, No. 4, pp. 708-715, 1986.

[7] S. Barbarossa, "Analysis of multicomponent LFM signals by a combined Wigner-Hough transform," IEEE Tran. Signal Processing, Vol. 43, No. 6, pp. 1511-1515, 1995.

[8] X.-G. Xia, "Discrete chirp-Fourier transform and its application to chirp rate estimation," IEEE Tran. Signal Processing, Vol. 48, No. 11, pp. 3122-3133, 2000.

[9] S. Peleg and B. Porat, "Linear FM signal parameter estimation from discrete-time observations," IEEE Trans. Aerosp. Electron. Syst., Vol. 27, No. 7, pp. 607-614, 1991.

[10] S. Peleg and B. Porat, "Estimation and classification of polynomial phase signals," IEEE Trans. Inform. Theory, Vol. 37, No. 3, pp. 422-430, 1991.

[11] S. Barbarossa, A. Scaglione, and G. Giannakis, "Product high-order ambiguity function for multicomponent polynomial phase signal modeling," IEEE Trans. Signal Processing, Vol. 48, No. 3, pp. 691-708,1998.

[12] B. Boashash and P. O'Shea, "Polynomial Wigner-Ville distributions and their relationship to time-varying higher order spectra," IEEE Trans. Signal Processing, Vol. 42, No. 1, pp. 216-220, 1994.

[13] B. Barkat and B. Boashash, "Design of higher order polynomial Wigner-Ville distributions," IEEE Trans. Signal Processing, Vol. 47, No. 9, pp. 2608-2611, 1999.

[14] P. O'Shea, "A new technique for estimating instantaneous frequency rate," IEEE Signal Processing Lett., Vol. 9, No. 8, pp. 251-252, 2002.

[15] J. J. Tuma, "Engineering mathematics handbook: Definitions, theorems, formulas, tables (2nd ed)," McGraw- Hill, 1979.

[16] B. Porat and B. Friedlander, "Asymptotic statistical analysis of the higher order ambiguity function for parameter estimation of the polynomial phase signal," IEEE Trans. Inform. Theory, Vol. 42, No. 5, pp. 995-1001, 1996.

[17] C. Ioana and A. Quinquis, "Time-frequency analysis using warped based high-order phase modeling," EURASIP Journal on Applied Signal Processing, Vol. 17, pp. 2856-2873, 2005.

[18] P.Wang and J. Yang, "Parameter estimation of multicomponent quadratic FM signals using computationally efficient Radon-CPF transform," Proc. of EUSIPCO, Florence, Italy, 2006.

[19] P. Wang, J. Yang, and I. Djurovic , "Algorithm extension of cubic phase function for quadratic FM signal," Proc. of ICASSP, Honolulu, USA, pp. 1125-1128, 2007.