Instantaneous Frequency Rate Estimation for High-Order Polynomial-Phase Signals

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Abstract—**Instantaneous frequency rate (IFR) estimation for high-order polynomial phase signals (PPSs) is considered. Specifically, an IFR estimator with only a second-order nonlinearity is proposed. The asymptotic mean-squared error (MSE) of the proposed IFR estimator is obtained via a multivariate first-order perturbation analysis. Our results show that the proposed estimator yields a smaller MSE and a lower signal-to-noise ratio (SNR) threshold than a popular IFR estimator involving higher nonlinearity. The proposed IFR estimator is also extended to estimate the phase parameters of a PPS. Numerical studies are presented to illustrate the performance of the proposed estimator.**

I. Introduction

Instantaneous frequency rate (IFR) reveals the rate-of-change of the frequency, which is proportional to the acceleration of a moving target [1]. Consider a polynomial-phase signal (PPS):

$$s(n) = Ae^{j\phi(n)} = Ae^{j\sum_{i=0}^{P} a_i n^i}, \qquad (1)$$

where P is the order of the PPS, A is the amplitude, $\phi(n)$ the instantaneous phase (IP) and $\{a_i\}_{i=0}^{P}$ the phase parameters, respectively. The IFR $\Omega(n)$ is defined as the second derivative of the IP [2]:

$$\Omega(n) = \frac{d^2\phi(n)}{dt^2} = \sum_{i=2}^{P} i(i-1)a_i n^{i-2}. \qquad (2)$$

The problem of interest is to estimate the IFR from noisy observations of $s(n)$. Three cases are of interest:

1. $P = 2$ (linear frequency-modulated (FM) signals): the IFR is often referred to as the chirp-rate, which can be estimated by well-established methods (e.g., [3] and references therein);
2. $P = 3$ (quadratic FM signals): the IFR is linearly proportional to the time n (cf. (2)), and can be estimated using the cubic phase function (CPF) [2];
3. $P > 3$ (high-order PPSs): the IFR is a nonlinear function of n, and can be estimated using the high-order phase function (HPF) [4].

For the high-order PPS, a qth-order HPF is defined as [4]

$$H_q(n,\omega) = \\ = \sum_{m=-M}^{M} \prod_{l=1}^{q/2} [s(n+d_l m)s(n-d_l m)]^{(r_l)} e^{-j\omega m^2},$$

where $2M+1$ is the length of a lag window, $\mathbf{d} \triangleq \{\mathbf{d_1}, \mathbf{d_2}, \cdots, \mathbf{d_{q/2}}\}$ denotes a set of lag-coefficients, $\mathbf{r} \triangleq \{\mathbf{r_1}, \mathbf{r_2}, \cdots, \mathbf{r_{q/2}}\}$ is used to impose complex conjugation if $r_l = -1$ or none if $r_l = 1$, and ω denotes the index in the IFR domain. The HPF order q and coefficient sets $\mathbf{d}$ and $\mathbf{r}$ are chosen such that the HPF is centered along the IFR of the signal [5, Proposition 1]. Note that the HPF with $q = 2$, $d_1 = 1$ and $r_1 = 1$ reduces to the CPF [2].

For high-order PPSs, the HPF often involves high-order nonlinearity. For example, a fourth-order PPS requires $q \geq 6$. Such high-order nonlinearity results in a large mean-squared error (MSE) and a high SNR threshold in IFR estimation. We herein propose an IFR estimator with only a second-order nonlinearity. Analytical results via a multivariate first-order perturbation analysis show that the proposed IFR estimator is asymptotically unbiased and provides a smaller MSE and a lower SNR threshold than the HPF-based approach.

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II. Proposed IFR Estimator

A. IFR Estimation for High-Order PPSs

Consider the phase of a bilinear transformation $s(n+m)s(n-m)$ for a Pth order PPS:

$$\begin{aligned}\arg\{s(n+m)\ s(n-m)\} &= \\ &= 2\phi(n) + \sum_{l=1}^{L} \frac{2\phi^{(2l)}(n)m^{2l}}{(2l)!} \\ &= 2\phi(n) + \Omega(n)m^2 + \sum_{l=2}^{L} \frac{2\phi^{(2l)}(n)m^{2l}}{(2l)!},\end{aligned}$$

where $L = \lfloor P/2 \rfloor$. Note that the resulting phase is a polynomial in m with even orders only, and each even order term is associated with a corresponding even-order derivative of the IP. In particular, the coefficient of the second-order term m^2 is the IFR $\Omega(n)$ of the signal.

In order to obtain these phase derivatives, a multidimensional matched filter can be applied:

$$A_{L,M}(n,\mathbf{\Psi}) = \\ = \sum_{m=-M}^{M} s(n+m)\, s(n-m)\, e^{-j\sum_{l=1}^{L}\omega_l m^{2l}}, \quad (3)$$

where $\mathbf{\Psi} \triangleq [\boldsymbol{\omega_1}, \boldsymbol{\omega_2}, \cdots, \boldsymbol{\omega_L}]^{\mathbf{T}}$ denotes the indices of the L even-order phase derivatives. Note that ω_1 is the IFR index for the proposed function in (3). When $L = 1$, the proposed function reduces to the CPF. In the absence of noise, the squared-magnitude of $A_{L,M}$ is maximized along the L phase derivatives, i.e., $\mathbf{\Psi_0} = \left[\mathbf{\Omega(n)}, \frac{\mathbf{2\phi^{(4)}(n)}}{\mathbf{4!}}, \cdots, \frac{\mathbf{2\phi^{(2L)}(n)}}{\mathbf{(2L)!}}\right]^{\mathbf{T}}$.

Now consider a noisy PPS $x(n) = s(n) + v(n)$, where $v(n)$ is a complex white Gaussian noise with zero mean and variance σ^2. The proposed estimator is given by

$$\left[\hat{\Omega}(n), \frac{2\hat{\phi}^{(4)}(n)}{4!}, \cdots, \frac{2\hat{\phi}^{(2L)}(n)}{(2L)!}\right]^T = \\ = \arg\max_{\mathbf{\Psi}} |B_{L,M}(n,\mathbf{\Psi})|^2, \quad (4)$$

where $B_{L,M}(n,\mathbf{\Psi})$ is defined similarly as $A_{L,M}(n,\mathbf{\Psi})$ in (3) with $s(n)$ replaced by $x(n)$.

Remark 1: The multidimensional matched filter in (3) is related to the local polynomial Fourier transform (LPFT) and the local polynomial Wigner distribution (LPWD) [6,7], but there are several notable differences. First, the LPFT and the LPWD were used to estimate the instantaneous frequency (IF) while our focus is IFR estimation. Second, the LPFT uses a P-dimensional matched filter while the proposed function (3) involves only a $\lfloor P/2 \rfloor$-dimensional matched filter. Third, the LPWD estimates the odd-order phase derivatives, while the proposed function estimates the even-order phase derivatives.

The proposed function (3) is similar to the CPF [4] in that both use the same second-order moment of the observed signal. As shown in Section II-C, the proposed function reduces to the CPF for a PPS with order less than 4. For higher-order PPSs, however, the CPF becomes inapplicable, whereas the proposed function can still offer statistically consistent IFR estimation.

B. Asymptotic Bias and MSE

The asymptotic bias and MSE of the estimator (4) are obtained. The detailed analysis is presented in the Appendix, which leads to the following result.

Proposition 1: For a Pth-order noisy PPS, the L phase-derivative estimates obtained in (4) are all asymptotically unbiased and their asymptotic MSEs are given by:

$$E\left\{(\delta\omega_l)^2\right\} = \\ = \frac{\left(1+\frac{1}{2\mathrm{SNR}}\right)}{4M^{4l+1}\cdot \mathrm{SNR}}\left[\mathbf{\Delta^{-1}}\right]_{ll}, l = 1,\cdots,L, \quad (5)$$

where $\mathbf{\Delta}$ is an $L \times L$ matrix with the lk-th element

$$[\mathbf{\Delta}]_{lk} = \frac{lk}{(2l+1)(2k+1)(2l+2k+1)}, \quad (6)$$

and the SNR is defined as A^2/σ^2.

From *Proposition 1*, the MSEs of the L estimates are independent of the phase parameters $\{a_i\}_{i=0}^{P}$ of the PPS. At high SNR, the MSEs are all approximately proportional to SNR^{-1}, while at low SNR they are proportional to SNR^{-2}. Moreover, the ith estimate, i.e., ω_i, has an MSE inversely proportional to M^{4i+1}. Hence, the larger the window length, the lower the MSE. As such, for a given SNR and time n, the asymptotic MSEs of the estimates are minimized by using the maximum window length given by 1) $M = \frac{N-1}{2} - |\frac{N-1}{2} - n|$ in the asymmetric sampling case with $n \in \{0, \cdots, N-1\}$; and 2) $M = \frac{N-1}{2} - |n|$ in the symmetric sampling case with $n \in \left\{-\frac{N-1}{2}, \cdots, \frac{N-1}{2}\right\}$, where N is the number of samples and we assume N is odd.

C. Illustrative Examples of $L = 1$ and $L = 2$

C.1 The PPS with order $P = 2$ and $P = 3$

Since $L = \lfloor P/2 \rfloor = 1$, $B_{L,M}(n, \mathbf{\Psi})$ reduces to

$$B_{1,M}(n, \omega_1) = \sum_m x(n+m)x(n-m)e^{-j\omega_1 m^2},$$

which is the CPF in [2]. In this case, the matrix $\mathbf{\Delta}$ reduces to a scalar 1/45. According to *Proposition 1* in the symmetric sampling case, the minimum MSE of the IFR estimate for a given SNR and time n is

$$E\left\{(\delta\omega_1)^2\right\} = \frac{45\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{4\left(\frac{N-1}{2} - |n|\right)^5 \cdot \mathrm{SNR}}. \quad (7)$$

which agrees with the result derived in (40) of [4].

C.2 The PPS with order $P = 4$ and $P = 5$

In this case, $B_{L,M}(n, \mathbf{\Psi})$ with $L = 2$ is

$$B_{2,M}(n, \omega_1, \omega_2) = \\ = \sum_m x(n+m)x(n-m)e^{-j\left(\omega_1 m^2 + \omega_2 m^4\right)}.$$

The estimates based on $|B_{2,M}|^2$ are $\hat{\omega}_1 = \hat{\Omega}(n)$ and $\hat{\omega}_2 = \hat{\phi}^{(4)}(n)/12$. According to *Proposition 1*, the MSEs of both ω_1 and ω_2 estimates are

$$E\left\{(\delta\omega_1)^2\right\} = 137.8\frac{\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{M^5 \cdot \mathrm{SNR}}, \quad (8)$$

$$E\left\{(\delta\omega_2)^2\right\} = 172.26\frac{\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{M^9 \cdot \mathrm{SNR}}. \quad (9)$$

For comparison, the MSEs of the HPF-based IFR estimator are proportional to SNR^{-6} due to a sixth-order nonlinearity ($q = 6$) [4, Section III], whereas the MSEs of the proposed IFR estimator are proportional to SNR^{-2} (cf. (8)) at low SNR. This implies that the proposed IFR estimator exhibits a much lower SNR threshold than the HPF-based method, which will be numerically verified in Section IV.

III. Estimation of Other Phase Parameters

As a by-product, the proposed estimator (4) can be utilized to estimate some phase parameters of a PPS. For example, consider a fourth-order PPS with phase $\phi(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + a_4 n^4$. The $B_{2,M}(n, \omega_1, \omega_2)$ in Section II-C.2 can be used to obtain two estimates, namely $\hat{\Omega}(n)$ in the ω_1 domain and $\hat{\omega}_2 = \hat{\phi}^{(4)}(n)/12 = 2\hat{a}_4$ in the ω_2 domain. The latter can be used to estimate a_4. The MSE of the a_4 estimate can easily be derived from (9):

$$E\left\{(\delta a_4)^2\right\} = \\ = \frac{E\left\{(\delta\omega_2)^2\right\}}{4}\Big|_{M=\frac{N-1}{2}} \approx \frac{22050}{N^9 \mathrm{SNR}}, \quad (10)$$

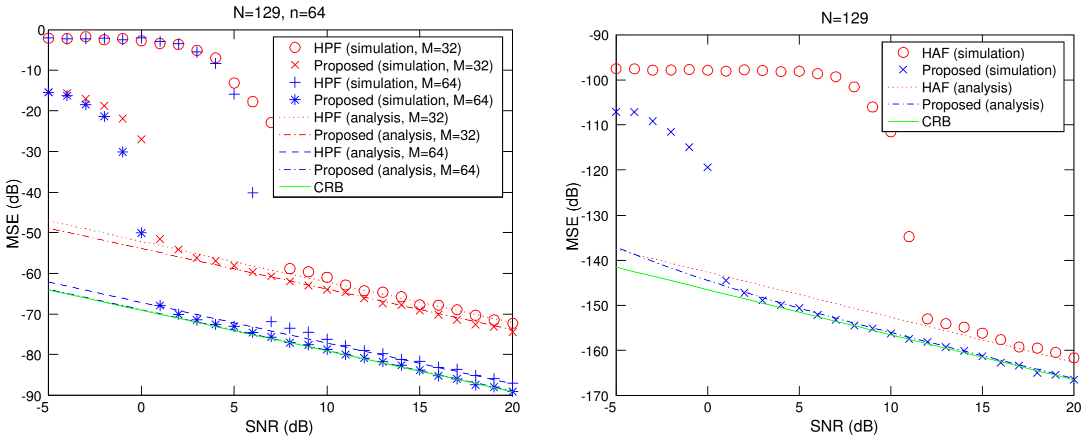
at high SNR. Meanwhile, the frequently used technique for a_4 estimation employs the high-order ambiguity function (HAF) [8]. The MSE of the HAF-based a_4 estimator is

$$E\left\{(\delta a_4)^2\right\}_{\mathrm{HAF}} \approx \frac{54351}{N^9 \mathrm{SNR}}, \quad (11)$$

at high SNR [8]. A comparison between (10) and (11) shows that our estimator provides a much lower MSE at high SNR. Moreover, since the Cramér-Rao bound (CRB) for a_4 is $\mathrm{CRB}\{a_4\} = 22050/(N^9 \mathrm{SNR})$ [9], the proposed a_4 estimator is asymptotically efficient at high SNR.

IV. Numerical Examples

Consider a fourth-order PPS with parameters $A = 1$, $(a_0, a_1, a_2, a_3, a_4) = \left(2, 0.02, 10^{-4}, 10^{-6}, 10^{-8}\right)$, and $N = 129$. The IFR is estimated at $n = 64$, which is the middle point of the observations. Fig. 1(a) shows



(a) (b)

Fig. 1. MSEs of parameter estimates for a fourth-order PPS, (a) IFR estimate; (b) Phase parameter a_4 estimate.

the MSEs obtained by analysis and simulation as a function of SNR for the proposed and, respectively, the HPF-based IFR estimators with two window sizes $M = 32$ and $M = 64$ (i.e., the maximum length at $n = 64$). We have the following observations: 1) At high SNR, the simulation agrees with the analysis for the MSE of the proposed estimator for both $M = 32$ and $M = 64$; 2) The MSE of the proposed estimator with $M = 64$ attains the CRB at high SNR; 3) With either window size, the MSE of the proposed estimator is generally lower than that of the HPF-based method at high SNR; 4) The proposed estimator gives a lower SNR threshold (of about 6 dB lower) than the HPF-based estimator.

We now consider the phase parameter a_4 estimation for the fourth-order PPS as explained in Section III. The MSE of the a_4 estimate is obtained at $n = 64$. The proposed estimator uses $B_{2,M}(n, \omega_1, \omega_2)$ with $M = 64$. The MSEs obtained by analysis and simulation are shown in Fig. 1(b). It is seen that, for SNR above 1 dB, the MSE obtained by simulation for the proposed a_4 estimator agrees with its theoretical result in (10). The MSE of the proposed estimator also reaches the CRB at high SNR. In addition, the SNR threshold is 11 dB for the HAF-based a_4 estimator, whereas it is 1 dB for the proposed a_4 estimator.

V. Conclusion

We have proposed an IFR estimator with a second-order nonlinearity. The asymptotic bias and MSE of the proposed estimator for a PPS with an arbitrary order have been obtained by using a multivariate first-order perturbation analysis. We have also discussed how to use the proposed estimator for the estimation of other phase parameters of a PPS. In particular, we showed that the a_4 estimator for a fourth-order PPS is asymptotically efficient at high SNR.

Appendix

[Asymptotic Bias and Variance] The asymptotic bias and variance of the estimator (4) are obtained using a multivariate first-order perturbation analysis which extends the univariate first-order perturbation analysis in [10]. For a noisy PPS $x(n) = s(n) + v(n)$, the $B_{L,M}(n, \mathbf{\Psi})$ can be decomposed into a noise-free component $C_s(n, \mathbf{\Psi})$ and a noisy component $C_{vs}(n, \mathbf{\Psi})$:

$$C_s(n, \mathbf{\Psi}) =$$

$$= \sum_{m=-M}^{M} s(n+n)s(n-m)e^{-j\sum_{l=1}^{L}\omega_l m^{2l}},$$

$$C_{vs}(n,\boldsymbol{\Psi}) = \sum_{\mathbf{m}=-\mathbf{M}}^{\mathbf{M}} \mathbf{z_{vs}}(\mathbf{n},\mathbf{m})\mathbf{e}^{-\mathbf{j}\sum_{\mathbf{l}=1}^{\mathbf{L}}\boldsymbol{\omega}_{\mathbf{l}}\mathbf{m}^{2\mathbf{l}}},$$

where $z_{vs}(n,m) = s(n+m)v(n-m) + s(n-m)v(n+m) + v(n+m)v(n-m)$. Let $f_{L,M}(n,\boldsymbol{\Psi})$ denote $|B_{L,M}(n,\boldsymbol{\Psi})|^2$, i.e., the objective function in (4). The $f_{L,M}(n,\boldsymbol{\Psi})$ can further be decomposed into $f_s(n,\boldsymbol{\Psi})$ and $f_{vs}(n,\boldsymbol{\Psi})$ within a first-order approximation [10]:

$$f_s(n,\boldsymbol{\Psi}) = C_s(n,\boldsymbol{\Psi})\mathbf{C}_\mathbf{s}^*(\mathbf{n},\boldsymbol{\Psi}) \quad (12)$$

$$f_{vs}(n,\boldsymbol{\Psi}) \approx 2\Re\left\{C_s(n,\boldsymbol{\Psi})\mathbf{C}_\mathbf{vs}^*(\mathbf{n},\boldsymbol{\Psi})\right\}, \quad (13)$$

where $\Re\{\cdot\}$ denotes the real part of $\{\cdot\}$. On one hand, from (3), the noise-free term $f_s(n,\boldsymbol{\Psi})$ is maximized at $\boldsymbol{\Psi}_\mathbf{0} = \left[\boldsymbol{\Omega}(\mathbf{n}), \frac{\mathbf{2}\phi^{(\mathbf{4})}(\mathbf{n})}{\mathbf{4}!}, \cdots, \frac{\mathbf{2}\phi^{(\mathbf{2L})}(\mathbf{n})}{(\mathbf{2L})!}\right]^\mathbf{T}$. On the other hand, the noisy term $f_{vs}(n,\boldsymbol{\Psi})$, acting like a random perturbation, moves the global maximum from $\boldsymbol{\Psi}_\mathbf{0}$ to $\boldsymbol{\Psi}_\mathbf{0} + \boldsymbol{\delta\Psi}$, where $\delta\boldsymbol{\Psi} \triangleq [\boldsymbol{\delta\omega_1}, \cdots, \boldsymbol{\delta\omega_L}]^\mathbf{T}$ is assumed to be small in an asymptotic sense. Our purpose here is to obtain the bias and variance of the L estimates.

Noting that $\boldsymbol{\Psi}_\mathbf{0}$ and $\boldsymbol{\Psi}_\mathbf{0} + \boldsymbol{\delta\Psi}$ are, respectively, the maxima of $f_s(n,\boldsymbol{\Psi})$ and $f_s(n,\boldsymbol{\Psi}) + f_{vs}(n,\boldsymbol{\Psi})$, we have

$$\frac{\partial f_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l} = 2\Re\left\{\frac{\partial C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l}C_s^*(n,\boldsymbol{\Psi}_0)\right\} = 0, \quad (14)$$

$$\frac{\partial f_s(n,\boldsymbol{\Psi}_0+\delta\boldsymbol{\Psi})}{\partial\omega_l} + \frac{\partial f_{vs}(n,\boldsymbol{\Psi}_0+\delta\boldsymbol{\Psi})}{\partial\omega_l} = 0, \quad (15)$$

where $l = 1,\cdots,L$. A first-order Taylor series expansion of (15) around $\boldsymbol{\Psi}_\mathbf{0}$ leads to

$$\frac{\partial f_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l} + \frac{\partial f_{vs}(n,\boldsymbol{\Psi}_0)}{\partial\omega_l} + \sum_{k=1}^{L}\frac{\partial^2 f_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l\partial\omega_k}\delta\omega_k = 0. \quad (16)$$

Due to (14), the first term of (16) is zero and, as a result, we can represent the above equations in a matrix form

$$\mathbf{f}_{vs} + \mathbf{F}_s\delta\boldsymbol{\Psi} = \mathbf{0}_{L\times1} \quad (17)$$

where

$$\begin{aligned}[\mathbf{F}_s]_{lk} &= 2\Re\left\{\frac{\partial^2 C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l\partial\omega_k}C_s^*(n,\boldsymbol{\Psi}_0) + \frac{\partial C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l}\frac{\partial C_s^*(n,\boldsymbol{\Psi}_0)}{\partial\omega_k}\right\}\\ &\overset{(a)}{=} -\frac{32A^4M^{(2l+2k+2)}lk}{(2l+1)(2k+1)(2l+2k+1)},\end{aligned} \quad (18)$$

and (a) here follows from the following result

$$\begin{aligned}C_s^*(n,\boldsymbol{\Psi}_0) &\approx \zeta A^2 2M,\\ \frac{\partial C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l} &= -j\zeta^*A^2\sum_{m=-M}^{M}m^{2l}\\ &\approx -j\zeta^*A^2\frac{2M^{(2l+1)}}{(2l+1)},\\ \frac{\partial^2 C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l\partial\omega_k} &= -\zeta^*A^2\sum_m m^{2l+2k}\\ &\approx -\zeta^*A^2\frac{2M^{(2l+2k+1)}}{(2l+2k+1)},\end{aligned}$$

with $\zeta \triangleq e^{-j2\phi(n)}$, and the approximation $\sum_{m=-M}^{M}m^{2k} \approx 2M^{(2k+1)}/2k+1$ if $M \gg 2k$. Furthermore,

$$\begin{aligned}[\mathbf{f_{vs}}]_l &= 2\Re\left\{\frac{\partial C_s(n,\boldsymbol{\Psi}_0)}{\partial\omega_l}C_{vs}^*(n,\boldsymbol{\Psi}_0) + C_s(n,\boldsymbol{\Psi}_0)\frac{\partial C_{vs}^*(n,\boldsymbol{\Psi}_0)}{\partial\omega_l}\right\}\\ &\overset{(a)}{=} -4A^2M\Im\{\Gamma(l)\},\end{aligned} \quad (19)$$

where $\Im\{\cdot\}$ denotes the imaginary part of $\{\cdot\}$,

$$\Gamma(l) \triangleq \zeta^*\sum_m\left(m^{2l} - \frac{M^{2l}}{(2l+1)}\right)z_{vs}^* e^{j\sum_{i=1}^{L}\frac{2\phi^{(2i)}(n)}{(2i)!}m^{2i}},$$

and (a) here follows from the following result

$$C_{vs}^{*}(n,\boldsymbol{\Psi}_0)=\sum_m z_{vs}^{*}e^{j\sum_{i=1}^{L}\frac{2\phi^{(2i)}(n)}{(2i)!}m^{2i}},$$

$$\frac{\partial C_{vs}^{*}(n,\boldsymbol{\Psi}_0)}{\partial\omega_l}=j\sum_m m^{2l}z_{vs}^{*}e^{j\sum_{i=1}^{L}\frac{2\phi^{(2i)}(n)}{(2i)!}m^{2i}}.$$

As a result, the error vector can be expressed as

$$\delta\boldsymbol{\Psi}=-\mathbf{F}_s^{-1}\mathbf{f}_{vs}. \tag{20}$$

Taking the expectation on both sides of (20) yields

$$E\left\{\delta\boldsymbol{\Psi}\right\}=-\mathbf{F}_s^{-1}E\left\{\mathbf{f}_{vs}\right\}=\mathbf{0}_{\mathbf{L}\times\mathbf{1}}, \tag{21}$$

since $E\left\{z_{vs}^{*}\right\}=s(n+m)E\left\{v\left(n-m\right)\right\}+s(n-m)E\left\{v\left(n+m\right)\right\}+E\left\{v\left(n+m\right)v\left(n-m\right)\right\}=0$ for any n and m [8] which results in $E\left\{\Gamma(l)\right\}=0$. Therefore, from (21), all L estimates are asymptotically unbiased.

The covariance matrix of the estimate from (20) is

$$E\left\{\delta\boldsymbol{\Psi}\delta\boldsymbol{\Psi}^{\mathbf{T}}\right\}=\mathbf{F}_s^{-1}\boldsymbol{\Xi}\mathbf{F}_{\mathbf{s}}^{-\mathbf{1}}, \tag{22}$$

where the diagonal elements give the variance of the L estimates, and the lk-th element of $\boldsymbol{\Xi}$ is

$$\begin{aligned}\boldsymbol{\Xi}_{\mathbf{lk}}&=E\left\{\left[\mathbf{f}_{\mathbf{vs}}\right]_l\left[\mathbf{f}_{\mathbf{vs}}\right]_k\right\}\\&\overset{(a)}{=}8A^4M^2\Re\left\{E\left[\Gamma(l)\Gamma^{*}(k)\right]-E\left[\Gamma(l)\Gamma(k)\right]\right\},\\&\overset{(b)}{=}\frac{128A^4\left(2A^2\sigma^2+\sigma^4\right)M^{2l+2k+3}lk}{(2l+1)\left(2k+1\right)\left(2l+2k+1\right)},\end{aligned} \tag{23}$$

where (a) follows from the fact that $E\left\{\Im\left[x\right]\Im\left[y\right]\right\}=0.5\Re\left[E\left\{xy^{*}\right\}-E\left\{xy\right\}\right]$, and (b) is due to

$$\begin{aligned}&E\left\{\Gamma(l)\Gamma^{*}(k)\right\}=\\&=\left(4A^2\sigma^2+2\sigma^4\right)\sum_m\left(m^{2l}-\frac{M^{2l}}{(2l+1)}\right)\\&\qquad\times\left(m^{2k}-\frac{M^{2k}}{(2k+1)}\right),\end{aligned} \tag{24}$$

$$E\left\{\Gamma(l)\Gamma(k)\right\}=0. \tag{25}$$

Eqn. (24) can be verified by using

$$\begin{aligned}&E\left\{z_{vs}^{*}\left(n,m_1\right)z_{vs}\left(n,m_2\right)\right\}=\\&=\left(2A^2\sigma^2+\sigma^4\right)\delta\left(m_1+m_2\right)\\&+\left(2A^2\sigma^2-\sigma^4\right)\delta\left(m_1-m_2\right)\end{aligned}$$

where $\delta(\cdot)$ denotes the Kronecker delta function. Finally, combining (18), (22) and (23) yields (5).

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