

Robust Time-Frequency Distributions

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Study of the additive Gaussian noise influence on time-frequency (TF) distributions is an important issue [Article 9.1]. However, in many practical applications, especially in communications, signals are disturbed by a kind of impulse noise. These noises are commonly modeled by heavy-tailed (long-tailed) probability density functions (pdfs) [6]. It is well known fact that the conventional TF distributions are quite sensitive to this kind of noise, which is able to destroy sensitive signal information. The minimax Huber M -estimates [5] can be applied in order to design the periodogram and TF distributions, robust with respect to the impulse noise. For nonstationary signals, the robust TF distributions are developed as an extension of the robust M -estimation approach.

A. Robust Spectrogram

The standard short-time Fourier transform (STFT), at a given point (t, f) , can be defined as a solution of the following optimization problem:

$$F_z(t, f) = \arg \left\{ \min_m I(t, f, m) \right\}, \quad (1)$$

$$I(t, f, m) = \sum_{n=-N/2}^{N/2-1} w(n\Delta t) \mathbf{F}(e(t, f, n)). \quad (2)$$

Here, the loss function is given as $\mathbf{F}(e) = |e|^2$, $w(n\Delta t)$ is a window function and Δt is a sampling interval. The error function has the form:

$$e(t, f, n) = z(t + n\Delta t)e^{-j2\pi f n\Delta t} - m, \quad (3)$$

where m is a complex-valued optimization parameter in (1). The error function can be considered as a residuum expressing the “similarity” between the signal and a given harmonic $\exp(j2\pi f n\Delta t)$.

Time-Frequency Signal Analysis and Processing, ed. B. Boashash, Elsevier, 2003.

The solution of (1) easily follows from

$$\frac{\partial I(t, f, m)}{\partial m^*} = 0 \quad (4)$$

in the form of the well-known standard STFT:

$$F_z(t, f) = \frac{1}{a_w} \sum_{n=-N/2}^{N/2-1} w(n\Delta t) z(t + n\Delta t) e^{-j2\pi f n\Delta t}, \quad (5)$$

where

$$a_w = \sum_{n=-N/2}^{N/2-1} w(n\Delta t). \quad (6)$$

The corresponding spectrogram is defined by

$$S_z(t, f) = |F_z(t, f)|^2. \quad (7)$$

The maximum likelihood (ML) approach can be used for selection of the appropriate loss function $\mathbf{F}(e)$ if the pdf $p(e)$ of the noise is known. The ML approach suggests the loss function $\mathbf{F}(e) \sim -\log p(e)$. For example, the loss function $\mathbf{F}(e) = |e|^2$ gives the standard STFT, as the ML estimate of spectra for signals corrupted with the Gaussian noise, $p(e) \sim \exp(-|e|^2)$. The standard STFT produces poor results for signals corrupted by impulse noise. Additionally, in many cases the ML estimates are quite sensitive to deviations from the parametric model and the hypothetical distribution. Even a slight deviation from the hypothesis can result in a strong degradation of the ML estimate. The minimax robust approach has been developed in statistics as an alternative to the conventional ML in order to decrease the ML estimates sensitivity, and to improve the efficiency in an environment with the heavy-tailed pdfs. The loss function

$$\mathbf{F}(e) = |e| = \sqrt{\Re^2\{e\} + \Im^2\{e\}} \quad (8)$$

is recommended by the robust estimation theory for a wide class of heavy-tailed pdfs. It is

worth noting that the loss function

$$\mathbf{F}(e) = |\Re\{e\}| + |\Im\{e\}| \quad (9)$$

is the ML selection for the Laplacian distribution of independent real and imaginary parts of the complex valued noise.

Nonquadratic loss functions in (1) can improve filtering properties for impulse noises. Namely, in [7], [8] it is proved that there is a natural link between the problem of spectra resistance to the impulse noise and the minimax Huber's estimation theory. It has been shown that the loss function derived in this theory could be applied to the design of a new class of robust spectra, inheriting properties of strong resistance to impulse noises.

In particular, the robust M -STFT has been derived by using the absolute error loss function $\mathbf{F}(e) = |e|$ in (1)-(4) [7]. It is a solution of the nonlinear equation:

$$F_z(t, f) = \frac{1}{a_w(t, f)} \times \sum_{n=-N/2}^{N/2-1} d(t, f, n) z(t + n\Delta t) e^{-j2\pi f n \Delta t}, \quad (10)$$

where:

$$d(t, f, n) = \frac{w(n\Delta t)}{|z(t + n\Delta t) e^{-j2\pi f n \Delta t} - F_z(t, f)|}, \quad (11)$$

and

$$a_w(t, f) = \sum_{n=-N/2}^{N/2-1} d(t, f, n). \quad (12)$$

If real and imaginary parts of the additive noise are independent, the statistically optimal robust estimation theory requires replacement of (1) with [8]:

$$F_z(t, f) = \arg \left\{ \min_m I_1(t, f, m) \right\}, \quad (13)$$

$$I_1(t, f, m) = \sum_{n=-N/2}^{N/2-1} w(n\Delta t) [\mathbf{F}(\Re\{e_1\}) + \mathbf{F}(\Im\{e_1\})] \quad (14)$$

where e_1 is an error function of the form

$$e_1(t, f, n) = z(t + n\Delta t) - m e^{j2\pi f n \Delta t}. \quad (15)$$

For $\mathbf{F}(e) = |e|$, the robust STFT (13) can be presented as a solution of (10), where $d(t, f, n)$ is given by:

$$d(t, f, n) = w(n\Delta t) \times \frac{|\Re\{e_1(t, f, n)\}| + |\Im\{e_1(t, f, n)\}|}{|\Re\{e_1(t, f, n)\}|^2 + |\Im\{e_1(t, f, n)\}|^2}. \quad (16)$$

The robust spectrogram defined in the form

$$S_z(t, f) = I_1(t, f, 0) - I_1(t, f, F_z(t, f)) \quad (17)$$

is called the residual spectrogram, in order to distinguish it from the amplitude spectrogram (7). For the quadratic loss function $\mathbf{F}(e)$ the residual spectrogram (17) coincides with the standard amplitude spectrogram (7). In [8] it has been shown that, in a heavy-tailed noise environment, the residual robust spectrogram performs better than its amplitude counterpart.

The accuracy analysis of the robust spectrograms, as well as a discussion on further details on the minimax approach, can be found in [7], [8].

B. Realization of the Robust STFT

B.1 Iterative Procedure

The expression (10) includes $F_z(t, f)$ on the right hand side. Therefore, to get the robust STFT we have to solve a nonlinear equation of the form $x = f(x)$. Here, we will use the fixed point iterative algorithm $x_i = f(x_{i-1})$, with the stopping rule $|x_i - x_{i-1}|/|x_i| < \eta$, where η defines the solution precision. This procedure, applied to (10), can be summarized as follows.

Step (0): Calculate the standard STFT (5): $F_z^{(0)}(t, f) = F_z(t, f)$, and $i = 0$.

Step (i): Set $i = i + 1$. Calculate $d^{(i)}(t, f, n)$ for $F_z^{(i-1)}(t, f)$ determined from (11) or (16). Calculate $F_z^{(i)}(t, f)$ as:

$$F_z^{(i)}(t, f) = \frac{1}{\sum_{n=-N/2}^{N/2-1} d^{(i)}(t, f, n)} \times \sum_{n=-N/2}^{N/2-1} d^{(i)}(t, f, n) z(t + n\Delta t) e^{-j2\pi f n \Delta t}. \quad (18)$$

Step (ii): If the relative absolute difference between two iterations is smaller than η :

$$\frac{|F_z^{(i)}(t, f) - F_z^{(i-1)}(t, f)|}{|F_z^{(i)}(t, f)|} \leq \eta, \quad (19)$$

then the robust STFT is obtained as $F_z(t, f) = F_z^{(i)}(t, f)$.

B.2 Vector Filter Approach

Note that the standard STFT (5) can be treated as an estimate of the mean, calculated over a set of complex-valued observations:

$$\mathbf{E}^{(t,f)} = \{z(t+n\Delta t)e^{-j2\pi fn\Delta t}: n \in [-N/2, N/2)\}. \quad (20)$$

If we restrict possible values of m in (1) to the set $\mathbf{E}^{(t,f)}$, the vector filter concept [1], [3], [9] can be applied to get a simple approximation of the robust estimate of the STFT. Here, the coordinates of vector-valued variable are real and imaginary parts of $z(t+n\Delta t)e^{-j2\pi fn\Delta t}$. The vector estimate of the STFT is defined as $F_z(t, f) = m$, where $m \in \mathbf{E}^{(t,f)}$, and for all $k \in [-N/2, N/2)$ the following inequality holds:

$$\begin{aligned} & \sum_{n=-N/2}^{N/2-1} \mathbf{F}(|m - z(t+n\Delta t)e^{-j2\pi fn\Delta t}|) \leq \\ & \sum_{n=-N/2}^{N/2-1} \mathbf{F}(|z(t+k\Delta t)e^{-j2\pi fk\Delta t} \\ & \quad - z(t+n\Delta t)e^{-j2\pi fn\Delta t}|). \end{aligned} \quad (21)$$

For $\mathbf{F}(e) = |e|$ this estimate is called the vector median.

The marginal median can be used for independent estimation of real and imaginary parts of $F_z(t, f)$. It results in

$$\begin{aligned} \Re\{F_z(t, f)\} &= \operatorname{median}_{n \in [-\frac{N}{2}, \frac{N}{2})} \{\Re\{z(t+n\Delta t)e^{-j2\pi fn\Delta t}\}\}, \\ \Im\{F_z(t, f)\} &= \operatorname{median}_{n \in [-\frac{N}{2}, \frac{N}{2})} \{\Im\{z(t+n\Delta t)e^{-j2\pi fn\Delta t}\}\}. \end{aligned} \quad (22)$$

The separate estimation of the real and imaginary parts of $F_z(t, f)$ assumes independence of the real and imaginary parts of $z(t+n\Delta t)e^{-j2\pi fn\Delta t}$, what in general does not hold here. However, in numerous experiments the accuracy of the median estimates (21) and (22) is of the same order. A simplicity of calculation is the advantage of these median estimates over the iterative procedures.

C. Robust Wigner Distribution

The standard Wigner distribution (WD) of a discrete-time signal is defined as:

$$\begin{aligned} W_z(t, f) &= \frac{1}{a_w} \\ & \times \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} w(n\Delta t) z(t+n\Delta t) z^*(t-n\Delta t) e^{-j4\pi fn\Delta t}, \end{aligned} \quad (23)$$

with the normalization factor

$$a_w = \sum_{n=-N/2}^{N/2} w(n\Delta t). \quad (24)$$

It can be interpreted as a solution of the problem

$$W_z(t, f) = \arg \left\{ \min_m J(t, f, m) \right\}, \quad (25)$$

$$\begin{aligned} J(t, f, m) &= \sum_{n=-N/2}^{N/2} [w(n\Delta t) \\ & \times \mathbf{F}(|z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t} - m|)], \end{aligned} \quad (26)$$

where $\mathbf{F}(e) = |e|^2$. For the loss function $\mathbf{F}(e) = |e|$, solution of (25)-(26) is a WD robust to the impulse noise. It can be obtained as a solution of the nonlinear equation [4]

$$\begin{aligned} W_z(t, f) &= \frac{1}{a_{we}(t, f)} \\ & \times \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} d(t, f, n) z(t+n\Delta t) z^*(t-n\Delta t) e^{-j4\pi fn\Delta t}, \end{aligned} \quad (27)$$

with

$$d(t, f, n) =$$

$$\frac{w(n\Delta t)}{|z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t} - W_z(t, f)|}, \quad (28)$$

$$a_{we}(t, f) = \sum_{n=-N/2}^{N/2} d(t, f, n).$$

An iterative procedure similar to the one described for the robust STFT can be used to find $W_z(t, f)$ from (27)-(28).

C.1 Properties of the Robust WD

1) The robust WD is real-valued for real and symmetric window function:

$$\begin{aligned} W_z^*(t, f) &= \frac{1}{a_{we}^*(t, f)} \\ &\times \sum_{n=-N/2}^{N/2} [w^*(n\Delta t)z^*(t+n\Delta t)z(t-n\Delta t) \\ &\times \frac{e^{j4\pi fn\Delta t}}{|z^*(t+n\Delta t)z(t-n\Delta t)e^{j4\pi fn\Delta t} - W_z^*(t, f)|}] \\ &= \frac{1}{a_{we}(t, f)} \sum_{n=-N/2}^{N/2} [w^*(-n\Delta t)z(t+n\Delta t)z^*(t-n\Delta t) \\ &\times \frac{e^{-j4\pi fn\Delta t}}{|z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t} - W_z^*(t, f)|}] \\ &= W_z(t, f). \end{aligned} \quad (29)$$

2) The robust WD is TF invariant. For signal $y(t) = z(t-t_0)e^{j2\pi f_0 t}$, we get $W_y(t, f) = W_z(t-t_0, f-f_0)$.

3) For linear FM signals $z(t) = \exp(jat^2/2 + jbt)$, when $w(n\Delta t)$ is very wide window, the WD is an almost ideally concentrated TF distribution.

C.2 Median WD

For rectangular window, the standard WD can be treated as an estimate of the mean, calculated over a set of complex-valued observations

$$\mathbf{G} = \{z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t} : n \in [-N/2, N/2]\}, \quad (30)$$

i.e.,

$$W_z(t, f) = \frac{1}{N+1}$$

$$\times \sum_{n=-N/2}^{N/2} z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t}. \quad (31)$$

From (29) follows that the robust WD is real-valued, thus the minimization of $J(t, f, m)$ can be done with respect to the real part of $z(t+n\Delta t)z^*(t-n\Delta t)e^{-j4\pi fn\Delta t}$ only. A form of the robust WD, the median WD, can be introduced as:

$$\begin{aligned} W_z(t, f) &= \underset{n \in [-\frac{N}{2}, \frac{N}{2}]}{\text{median}} \{ \Re\{z(t+n\Delta t) \\ &\times z^*(t-n\Delta t)e^{-j4\pi fn\Delta t}\} \}. \end{aligned} \quad (32)$$

Generally, it can be shown that any robust TF distribution, obtained by using the Hermitian local auto-correlation function (LAF), $R_z(t, n\Delta t) = R_z^*(t, -n\Delta t)$ in the minimization, is real-valued. In the WD case this condition is satisfied, since $R_z(t, n\Delta t) = z(t+n\Delta t)z^*(t-n\Delta t)$. For a general quadratic distribution from the Cohen class with a Hermitian LAF, the proposed robust version reads

$$\begin{aligned} \rho_z(t, f) &= \\ &\underset{n \in [-\frac{N}{2}, \frac{N}{2}]}{\text{median}} \{ \Re\{R_z(t, n\Delta t)e^{-j4\pi fn\Delta t}\} \}, \end{aligned} \quad (33)$$

where $R_z(t, n\Delta t)$ includes the kernel in time-lag domain.

Note, that for an input Gaussian noise the resulting noise in the WD has both Gaussian and impulse component, due to the WD's quadratic nature. Thus, as it is shown in [2], robust WD forms can improve performance of the standard WD, even in a high Gaussian input noise environment.

D. Example

Consider the nonstationary FM signal:

$$z(t) = \exp(j204.8\pi t|t|), \quad (34)$$

corrupted with a high amount of the heavy-tailed noise:

$$\varepsilon(t) = 0.5(\varepsilon_1^3(t) + j\varepsilon_2^3(t)), \quad (35)$$

where $\varepsilon_i(t)$, $i = 1, 2$ are mutually independent Gaussian white noises $\mathcal{N}(0, 1)$. We consider the interval $t \in [-7/8, 7/8]$ with a sampling rate $\Delta t = 1/512$ for spectrograms, and

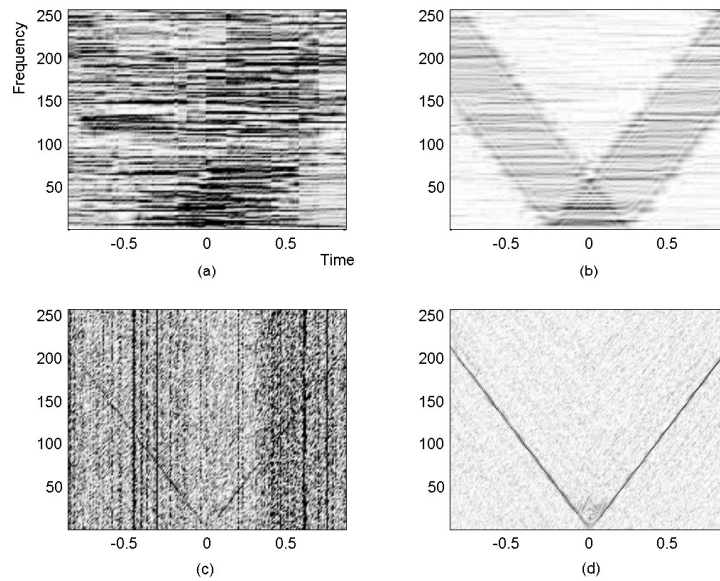


Fig. 1. TF representations of signal corrupted by impulse noise: a) Standard spectrogram; b) Robust spectrogram; c) Standard WD; d) Robust WD.

$\Delta t = 1/1024$ for WDs. The rectangular window width is $N = 256$ in all cases. The standard spectrogram and the WD (Figs.1(a),(c)) are calculated according to (5) and (23). The robust spectrogram (Fig.1(b)) is calculated by using iterative procedure (18)-(19). In this case, similar results would be produced by residual spectrogram (13)-(17), vector median (21), and marginal median (22). The robust WD (Fig.1(d)) is calculated by using expression (32) for the considered TF point. It can be concluded from Fig.1 that the robust spectrogram and the robust WD filter the heavy-tailed noise significantly better than the standard spectrogram and the standard WD. Note that the standard and the robust WD exhibit higher TF resolution in comparison with the corresponding spectrograms.

E. Summary and Conclusions

The TF distributions are defined within the Huber robust statistics framework. The loss function $\mathbf{F}(e) = |e|$ gives distributions robust to the impulse noise influence. They can be realized by using: the iterative procedures, the vector median, or the marginal median approach. All calculation procedures produce ac-

curacy of the same order of magnitude.

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