An Analysis of Some Time-Frequency and Time-Scale Distributions

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Abstract— **This paper presents an analysis of the representation of instantaneous frequency and group delay using time-frequency transforms or distributions of energy density domain. The time frequency distributions which ideally represent the instantaneous frequency or group delay (ITFD) are defined. Closeness to the ITFD is chosen as a criterion for comparison of various commonly used distributions. It is shown that the Wigner distribution is the best among them, with respect to this criterion. The wavelet and scaled forms of the Wigner distribution are defined and analyzed. In the second part of the paper we extended the analysis to the multicomponent signals and cross terms effects. On the basis of that analysis, an efficient method, derived from the analysis of the Wigner distribution defined in the frequency domain, is proposed. This method provides some substantional advantages over the Wigner distribution. The theory is illustrated on numerical examples.**

I. INTRODUCTION

Time-frequency distributions have been intensively studied during the past decade. We refer to several excellent review papers on the distributions for time-frequency analysis [1],[2],[3]. The commonly used energy density domain (E-domain, [6]) distributions are the following: Spectrogram (the squared modulus of the Short Time Fourier Transform), Scalogram (the squared modulus of the Wavelet transform), Wigner distribution and its variations - Generalized Wigner Distribution - GWD [5],[6].

It is desirable that an energetic time-frequency distribution (TFD) of a signal $x(t)$ satisfies the following basic properties:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_x(\omega,t)d\omega dt = E_x \qquad (1)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_x(\omega,t)d\omega = \mid x(t) \mid^2 \qquad (2)$$

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$$\int_{-\infty}^{\infty} P_x(\omega,t)dt = \mid X(\omega) \mid^2 \qquad (3)$$

where E_x and $X(\omega)$ denote the energy and the Fourier transform of $x(t)$, respectively, [1],[2],[3],[26]. It is obvious that if either of marginal properties (2), (3) is fulfilled, so is property (1). Note that an infinite number of distributions satisfying (1), (2) and (3) can be defined - the Cohen class of distributions [1]. This class of distributions has been derived in the quantum mechanics using characteristic functions approach. That derivation may be found in [1]. In Appendix A we provided a very simple proof of the following statement: If we know only one distribution satisfying (1),(2) and (3), an infinite number of distributions which satisfy the same conditions exist.

The relations (1),(2) and (3), do not tell anything about the local distribution of energy, at a point (ω, t). The concept of "time-frequency energy density at every point in the time-frequency plane is a priori impossible since the uncertainty principle does not allow the notion of energy at a specific time and frequency" [2],[9].

In this paper, we will impose some more specific requirements than the ones given by (2) and (3). Those requirements will turn out very reasonable and meaningful for specific classes of signals (belonging to the class of asyptotic signals), both monocomponent and multicomponent.

In Section II we define the ideal time frequency distribution with respect to the instantaneous frequency and group delay. In the sections that follow the commonly used distributions of E-domain are compared to the "ideal" ones. The modifications of the WD, the wavelet WD and the scaled WD, are proposed and analyzed. The multicomponent signals and cross terms effects ([10],[11],[24]) are studied next. A method for cross term reduction is proposed.

II. A DEFINITION OF THE IDEAL TIME-FREQUENCY DISTRIBUTION

Consider a complex signal $x(t)$ defined by:

$$x(t) = g(t)e^{j\phi(t)} \tag{4}$$

If we want to compare various distribution, what is the principle aim of this paper, we have to give an answer to the question: How does the ideal time-frequency representation of $x(t)$ look like? Generally, when $g(t)$ and $\phi(t)$ are arbitrary functions the answer is difficult and depends on our particular expectation from the time-frequency representation. Various distributions (Wigner distribution, Rihaczek distribution, Page distribution, Choi-Williams distribution,... a complete list may be found in [2]) have been derived, starting from the different prepositions.

In this paper, we will restrict ourselves to the class of signals having constant or slow-varying $g(t)$ as compared to the variation of $\phi(t)$[1]. This class of signals is practically important and, as it will be seen, permits a relatively simple definition of the ideal time-frequency representation.

[1]Signal $x(t)$ satisfying the condition that $g(t)$ is slow-varying comparing to the variation of $\phi(t)$ belongs to the class of asymptotic (sophisticated) signals.

If one may assume that the signal energy is approximately localized in the band $[-B/2, B/2]$ and in the time interval $[-T/2, T/2]$, then the signal is asymptotic if BT is large ($BT >> 1$), [18],[7].

The classical definitions of the durations T and B are:

$$T = \frac{\int_{-\infty}^{\infty} t^2|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \qquad B = \frac{\int_{-\infty}^{\infty} \omega^2|X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

For signal $x(t)$ defined by (4) we have:

$$\int_{-\infty}^{\infty} \omega^2 \mid X(\omega) \mid^2 d\omega =$$

$$= 2\pi \int_{-\infty}^{\infty} \mid \left(\frac{dg(t)}{dt}\right)^2 + g^2(t)\left(\frac{d\phi(t)}{dt}\right)^2 \mid dt$$

It folows that T depends only on $g(t)$, while B depends on both $g(t)$ and $\phi(t)$. Having $\phi(t)$ rapidly varying function comparing to $g(t)$ (i.e. $\phi'(t) >> g'(t)$) the value of B, as well as the product BT can be made arbitrary large. Thus, the signal $x(t)$ is an asymptotic one.

The stationary phase method [18],[7],[19] [2], applied to the Fourier analysis states that the value of the Fourier transform $X(\omega)$:

$$X(\omega) = \int_{-\infty}^{\infty} g(t)e^{j\phi(t)}e^{-j\omega t}dt \tag{5}$$

is approximately equal to:

$$X(\omega) \cong x(t_0)e^{-j\omega t_0}\sqrt{2\pi j/\phi''(t_0)} \tag{6}$$

provided that $g(t)$ is slow-varying, i.e. that $g'(t) << \phi'(t)$. The instant t_0, called the stationary phase point, is obtained from:

$$\frac{d}{dt}[\phi(t) - \omega t] = 0_{|t=t_0} \Leftrightarrow \omega = \phi'(t_0) \tag{7}$$

Equations (6) and (7) mean that the spectral component of signal $x(t)$ at a given ω is determined by the value of $x(t)$ at the instant t_0. This may be interpreted in the following way: The spectral component corresponding to the instant t is located at the frequency ω equal to the instantaneous one at the instant $t, \omega = \phi'(t)$.

The previous analysis leads to the definition of the Ideal time-frequency distribution (ITFD) in the form:

$$ITFD_x^{\omega}(\omega, t) = 2\pi \left|g(t)\right|^2 \delta[\omega - \phi'(t)] \tag{8}$$

where:

$$\omega_i(t) = \frac{d\phi(t)}{dt} \equiv \phi'(t)$$

denotes the instantaneous frequency of $x(t)$. The existence of $\phi'(t)$ is assumed. In other words, we require that the "Ideal TFD" has the instantaneous power $\mid g(t) \mid^2$ concentrated at the instantaneous frequency $\omega_i(t)$.

[2]If the function $g(t)$ is continuous and the derivative of the function $\mu(t)$ vanishes at only a single point $t = t_0$ in the interval $(-\infty, \infty)$:

$$\mu'(t_0) = 0 \text{ and } \mu''(t_0) \neq 0$$

then, for sufficiently large k,

$$\int_{-\infty}^{\infty} g(t)e^{jk\mu(t)}dt$$

$$\cong e^{jk\mu(t_0)}g(t_0)\sqrt{2\pi j/[k\mu''(t_0)]}$$

The proof may be found in [7].

If eqn. (7) has multiple solutions t_0, the results may be easily generalized. Physically, it means that the energy is concentrated on the same frequency at more than one instant. Note, if the condition $g'(t) \ll \phi'(t)$ is not satisfied, then the instantaneous bandwidth [28], rather than the instantaneous frequency, should be considered. However, the analysis in that case is more cumbersome and will not be pursued in this paper.

It is apparent that the distribution defined by (8) satisfies properties (1) and (2). The property (3) is also fulfilled under certain conditions[3] .

If the signal $x(t)$ is real, expressed by $x(t) = g(t)\cos(\phi(t))$, then the instantaneous frequency of $x(t)$ will be defined as [20]:

$$\omega_i(t) = \frac{d}{dt}[\arg z(t)]$$

where $z(t)$ is the analytic part (which may be written in form (4)) defined as:

$$z(t) = x(t) + jH[x(t)]$$

with $H[\ ]$ denoting the Hilbert transform operator.

Another class of signals, that will be considered, is the one whose Fourier transform $X(\omega)$ can be written as:

$$X(\omega) = G(\omega)e^{j\varphi(\omega)} \tag{9}$$

where $G(\omega)$ is slow-varying comparing to $\varphi(\omega)$. This signal belongs to the class of asymptotic signals, as well. All the above cosiderations for signal (4) are valid for (9) in the dual sense.

The group delay, as a dual notion of the instantaneous frequency, is defined by $t_g(\omega) = -\varphi'(\omega)$. The "Ideal TFD", in this case, should have the spectral energy density $\mid G(\omega) \mid^2$ located at its group delay:

$$ITFD_x^t(\omega,t) = \mid G(\omega) \mid^2 \delta[t + \varphi'(\omega)] \tag{10}$$

In the ensuing sections we will compare the commonly used time-frequency representations with the "ideal" ones, defined by (8) and (10). The measure of quality will be the similarity to $\text{ITFD}_x^\omega(\omega,t)$ or $\text{ITFD}_x^t(\omega,t)$, depending on the behavior of the signal $x(t)$.

III. INSTANTANEOUS FREQUENCY REPRESENTATION

A. Short Time Fourier Transform - STFT

The oldest time-frequency signal representation is based on the use of Short time Fourier transform (STFT) (or Running Fourier transform), [2],[3],[7],[8]:

$$STFT(\omega,t) = \int_{-\infty}^{\infty} x(\tau)w^*(\tau - t)e^{-j\omega(\tau-t)}d\tau \tag{11}$$

where $w(t)$ is a window, which is usually a real and even function.

For the signal $x(t)$ defined by (4), we have:

$$STFT(\omega,t) = \int_{-\infty}^{\infty} g(t+\tau)e^{j\phi(t+\tau)}w(\tau)e^{-j\omega\tau}d\tau \tag{12}$$

Expanding $\phi(t+\tau)$ into a Taylor series and assuming that the variations of $g(t+\tau)$, inside the window, are not significant, $g(t+\tau)w(\tau) \cong g(t)w(\tau)$, we get:

$$\mid STFT(\omega,t) \mid^2 = \mid g(t) \mid^2 \mid \frac{1}{2\pi}\delta[\omega - \phi'(t)]$$

$$*_\omega W(\omega) *_\omega \mathcal{F}[e^{j\phi''(t+\tau_1)\tau^2/2!}] \mid^2 \tag{13}$$

with: $\mathcal{F}[]$-the Fourier transform (FT) operator, $W(\omega) = \mathcal{F}[w(t)]; \tau_1$−a variable in the interval $[0,\tau]; *_\omega$−a convolution in ω. The continuity of the functions $\phi(t+\tau), \phi'(t+\tau)$ and $\phi"(t+\tau)$ inside the window $w(\tau)$ is assumed. This is the condition for the expansion of $\phi(t+\tau)$ into a Taylor series up to the second order term.

From eqn. (13) we see that the squared modulus of the STFT (spectrogram) is of the form similar to (8) convolved with the Fourier

[3] Note that

$$\int_{-\infty}^{\infty} \mid g(t) \mid^2 \delta[\omega - \phi'(t)]dt$$

$$= \mid g(t_0) \mid^2 \mid \frac{1}{\phi''(t_0)} \mid = \frac{1}{2\pi} \mid X(\omega) \mid^2$$

for $\phi'(t_0) = \omega$ and if $g(t)$ is sufficiently smooth. The substitution of variables $\phi'(t) = u$ (with $\phi''(t)dt = du$) and equation (6) is used.

transform of window and higher order terms of the derivatives of $\phi(t)$, starting with the second order term. If the phase $\phi(t)$ is not linear, then the artifacts are due to the second order term, thus being rather significant, because the weighting coefficient for the second order term, in the Taylor series expansion, is significant ($a_2 = 1/2!$).

If $\phi(t)$ is a linear function then the spectrogram is of the form $|STFT(\omega,t)|^2 = |g(t)|^2 \left|W(\omega - \phi'(t)\right|^2$.It is evident that $|STFT(\omega,t)|^2$ in this case behaves as $ITFD_x^\omega(\omega,t)$ if $| W(\omega) |^2$ is close to $2\pi\delta(\omega)$.

B. Wavelet Transform - WT

The definition of a continuous wavelet transform is:

$$WT(a,t) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} x(\tau) h^*(\frac{\tau - t}{a}) d\tau \quad (14)$$

where the scale a is used instead of frequency ($a = \omega_0/\omega$); $h(t)$ is a band-pass signal [2],[8]. We will choose $h(t)$ in the form $h(t) = w(t)e^{j\omega_0 t}$ which provides a strong formal connection of the WT with the STFT, i.e.:

$$WT(a,t) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} x(\tau)$$

$$\times w^*(\frac{\tau - t}{a}) e^{-j\omega_0 \frac{\tau - t}{a}} d\tau \quad (15)$$

where ω_0 is a constant and $w(t)$ is a window as in (11).

After some modifications following section 3.1, we get:

$$| WT(a,t) |= \frac{1}{2\pi} \sqrt{|a|} \mid g(t) \mid\mid \delta[\theta - a\phi'(t)]$$

$$*_\theta W(\theta) *_\theta \mathcal{F}[e^{j\phi''(t+a\tau_1)(a\tau)^2/2!}]|_{/\theta=\omega_0} \quad (16)$$

Let us, for sake of simplicity assume that the most significant disturbing term is of the second order, while the higher ones can be neglected, i.e. $\tau_1 = 0$. In this case we have:

$$| WT(a,t) |= \frac{1}{2\pi} \sqrt{|a|} \mid g(t) \mid$$

$$\times \mid \delta[\theta - a\phi'(t)] *_\theta W(\theta)$$

$$*_\theta \sqrt{\frac{2\pi j}{a^2\phi''(t)}} e^{-j \frac{\theta^2}{2a^2\phi''(t)}}] \mid_{/\theta=\omega_0} \quad (17)$$

For $a^2\phi"(t) \to 0$ the function describing the artifacts tends to $2\pi\delta(\theta)^4$, so we have:

$$| WT(a,t) |^2 = | a || g(t) |^2 | W[\omega_0 - a\phi'(t)] |^2$$

or

$$| WT(a,t) |^2 = | a || g(t) |^2 | W[a(\omega - \phi'(t))] |^2 . \quad (18)$$

The scalogram ($| WT(a,t) |^2$) is concentrated around the instantaneous frequency. If a is increased the concentration in (18) is better, but at the same time, if $\phi"(t)$ (or any higher order derivative) is not zero, then the factor $a^2\phi"$(t) increases, thus making the contribution of the artifacts significant, eqns.(16), (17). This means that the scalogram converges to the ITFD$_x^\omega$ only if a is large and if, at the same time, the frequency is a constant (the phase is linear). When that is not the case, the wavelet transform will nowhere produce the ideal concentration. This may be observed from the numerical example with the signal of the form $x(t) = Ae^{j\beta t^2}$, Fig.2d. For a small a the width of $W(\theta)$ is large, so the resolution is low, and for a great a the window $w(\tau/a)$ is wide, so the influence of the instantaneous frequencies variation is very emphasized, thus the resolution is low again.

C. Generalized Wigner Distribution - GWD

A very often used distribution of E-domain is the Wigner distribution (WD):

$$WD(\omega,t) = \int_{-\infty}^{\infty} x(t+\tau/2)x^*(t-\tau/2)e^{-j\omega\tau} d\tau \quad (19)$$

whose pseudo generalized form (PGWD) is:

$$PGWD(\omega,t) = \int_{-\infty}^{\infty} x(t + (\alpha + 1/2)\tau)$$

$$\times x^*(t + (\alpha - 1/2)\tau) w_s(\tau) e^{-j\omega\tau} d\tau \quad (20)$$

[4] A posible definition of delta function is [7]:

$$\delta(t) = \lim_{c \to o} \frac{1}{\sqrt{j\pi c^2}} e^{jt^2/c^2}$$

where α is a constant and $w_s(\tau) = w(\tau/2)w^*(-\tau/2)$.

For $\alpha = 0$ we get the WD (or the pseudo WD) and for $\alpha = 1/2$ the Rihaczek distribution (RD), [1],[2].

Expanding $\phi(t+(\alpha+1/2)\tau)$ and $\phi(t+(\alpha-1/2)\tau)$ into a Taylor series, about t we get:

$$PGWD(\omega,t) = \frac{1}{2\pi} \mid g(t) \mid^2$$
$$\times\delta[\omega - \phi'(t)] *_\omega W_s(\omega)$$
$$*_\omega \mathcal{F}[e^{j\phi''(t)\alpha\tau^2} e^{j\phi'''(t+\tau_1)\tau^3(1+12\alpha^2)/24}] \quad (21)$$

It is apparent that $PGWD(\omega,t)$ is a convolution of the ITFD_x^ω, the FT of the window and the FT of a function of the higher order derivatives of $\phi(t)$.A very interesting case is for the value $\alpha = 0$ (WD), for which the second order term is zero, as well as all the even-order terms. The lowest disturbing term is of the third order (the weighting coefficient for the third order term, in the Taylor series expansion, is $a_3 = 1/3!$). In other words, the WD is closer to the ITFD_x^ω than any other GWD for $\alpha \neq 0.$ If a signal is linear frequency modulated, i.e. $\phi(t) = at + bt^2$, then the WD, for a wide $w(t)$, is identical to the ITFD_x^ω. It means that the PWD allows a quadratic phase variation (inside a window) without artifacts.

As it is known, all time-frequency distributions satisfying the marginal properties, belong to the general Cohen class of distributions [1]:

$$CD(t,\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} x(u+\tau/2)$$
$$\times x^*(u-\tau/2)c(\theta,\tau)e^{-j\theta t - j\tau\omega + j\theta u} du d\tau d\theta \quad (22)$$

where $c(\theta,\tau)$ is a kernel function. The marginal properties (2) and (3) are satisfied if $c(\theta,0) = 1$ and $c(0,\tau) = 1$, (see Appendix A).

The question is now whether there exist another distribution, besides the WD, producing the ideal representation of the signal with a phase of the form $\phi(t) = a + a_1 t + a_2 t^2$? Substituting the signal $x(t) = ge^{j(a+a_1t+a_2t^2)}$ into eqn. (20) and equating the result with (8), it turns out that the kernel has to be $c(-2a_2\tau,\tau) = 1$ for all a_2 and τ. Thus $c(\theta,\tau) = 1$, for all θ and τ. That is just the kernel for the Wigner distribution. This leads to the conclusion that only the Wigner distribution, out of the Cohen class, produces the ideal representation of instantaneous frequency for linear frequency variation inside the window. This consideration is done (and it is valid), assuming signal independent kernels.

D. Wavelet Wigner Distribution - WWD

We can define the wavelet form of the Wigner distribution, which will be related to the WD in the same manner as the WT is to the STFT. This definition will have a frequency varying resolution (like the WT) and the artifacts will appear only if the variation of phase is of the higher than second order.

$$PWWD(a,t) = \int\limits_{-\infty}^{\infty} x(t+\tau/2)$$
$$\times x^*(t-\tau/2)w_s(\tau/a)e^{-j\tau\omega_0/a}d\tau \qquad (23)$$

This distribution belongs to the general class of the Affine smoothed pseudo Wigner distributions[5] defined in [22], as:

$$\Omega(a,t) = \frac{1}{|a|}\int_{-\infty}^{\infty} h(\frac{\tau}{a})[\int_{-\infty}^{\infty} g(\frac{u-t}{a})\times$$
$$x(u+\tau/2)x^*(u-\tau/2)du]e^{-\tau\omega_0/a}d\tau \qquad (24)$$

[5]The Affine class of distributions is the one preserving the time shift and time scaling:

$$x_1(t) = \sqrt{|a|}x[a(t-t_0)] \Rightarrow P_{x_1}(\omega,t) = P_x[a(t-t_0),\omega/a]$$

Any time-frequency distribution belonging to the class of Affine distribution (AD) may be derived from the Wigner distribution, [21],[22],[2]:

$$AD(a,t) = \frac{1}{2\pi}\int\int WD(\tau,\theta)\Pi(\frac{\tau-t}{a},a\theta)d\tau d\theta$$

where Π is an arbitrary function. For scalogram this function is equal to the Wigner distribution of the vasic wavelet $h(t)$. The previous formula is a cuonterpart of the Cohen class of distributions written in the form (eqn. (A.7)):

$$CD(\omega,t) = \frac{1}{2\pi}\int\int WD(\tau,\theta)\Pi(\tau-t,\theta-\omega)d\tau d\theta$$

The kernel $c(\theta,\tau)$ in (20) is a two-dimensional Fourier transform of $\Pi(t,\omega)$.

Taking $g((u-t)/a) = \delta((u-t)/a)$ and $h(\tau/a) = w_s(\tau/a)$ we get (21). The influence of smoothing along the frequency axis will be considered later, in the section dealing with multicomponent signals.

For the signal defined by (4) we obtained:

$$PWWD(a,t) = \frac{1}{2\pi} \mid a \mid\mid g(t) \mid^2 \delta[\phi'(t)a - \theta]$$

$$*_\theta W_s(\theta) *_\theta \mathcal{F}[e^{j\phi'''(t+a\tau_1)(a\tau)^3/24}]_{/\theta=\omega_0} \quad (25)$$

Assuming that the terms of the 5th and higher orders, in the Taylor series of $\phi(t+\tau)$, can be neglected, we have:

$$PWWD(a,t) = \frac{1}{2\pi} \mid a \mid\mid g(t) \mid^2 \delta[\phi'(t)a - \theta] *_\theta$$

$$W_s(\theta) *_\theta \sqrt{\frac{\pi j}{\sqrt{a^3\phi'''(t)/4}}} e^{-j0.385\theta\sqrt{\frac{12\theta}{a^3\phi'''(t)}}}]_{/\theta=\omega_0}$$

If $\phi'''(t)a^3/12$ is a small number then PWWD$(a,t) = \mid a \mid\mid g(t) \mid^2 \times W_s[\phi'(t)a - \omega_0]$, which is the shape similar to (18) in the WT. But the artifacts do not appear in this case if the phase variation is up to the second order. For large a (small frequencies) the concentration is close to the ideal one if the frequency is constant or linear, Fig.3e. For small a (high frequencies) the artifacts (mainly depending on $a^3\phi'''(t)/12$) are decreased.

E. Scaled Wigner Distribution - LWD

The decrease of artifacts in the WWD, for small a, leads to the idea of decreasing the artifacts at all frequencies. We can define such a distribution which is closer to $ITFD_x^\omega(\omega,t)$ than the WD. One distribution which significantly reduces the artifacts, independently from the frequency, is introduced as pseudo L-Wigner distribution (PLWD):

$$PLWD(\omega,t) = \int_{-\infty}^{\infty} x^L(t + \frac{\tau}{2L})$$

$$\times x^{*L}(t - \frac{\tau}{2L}) w_s(\tau) e^{-j\omega\tau} d\tau \quad (26)$$

where L is any integer greater than 0.

The PLWD is:

$$PLWD(\omega,t) = \frac{1}{2\pi} \mid g(t) \mid^{2L} \delta[\omega - \phi'(t)]$$

$$*_\omega W_s(\omega) *_\omega \mathcal{F}[e^{j2\frac{\phi'''(t+\tau_1)\tau^3}{2^3L^{3-1}3!}}] \quad (27)$$

This form has the artifacts of the third and higher orders (the odd ones), which are divided by the factor L^{n-1}. For example, for $L=2$, the dominant third term is divided by 4 (which is equivalent to $12dB$). This produces a significant improvement with respect to the WD. The PLWD produces a very high concentration of the generalized power $\mid x(t) \mid^{2L}$ at the instantaneous frequency $\phi'(t)$. If the instantaneous frequency is linear function of time, then the WD produces the ideal concentration. But, if that is not the case, then $L > 1$ dramatically reduces the distorsion. In other words, the L-Wigner distribution locally linearizes the instantaneous frequency function. In Section 5. and Appendix B, it is shown that the L-Wigner distribution may be efficiently realized, without need for oversampling, using recusive formulae.

The properties of the PLWD, equivalent to the ones of the PWD [2],[3], may be easily derived, [12]. The L-Wigner distribution may also be derived as a special and optimal case of the Wigner higher order spectra [23],[29].

IV. GROUP DELAY REPRESENTATION

Group delay, as a dual notion of the instantaneous frequency, can be analyzed from the definitions of the time-frequency distributions or transforms in the frequency domain [1],[2].

A. Short Time Fourier Transform

The definition of STFT, (11), in the frequency domain is:

$$STFT(\omega,t) = \frac{e^{j\omega t}}{2\pi} \int_{-\infty}^{\infty} X(\theta+\omega) W^*(\theta) e^{j\theta t} d\theta \quad (28)$$

For the signal $x(t)$ whose Fourier transform $X(\omega)$ is of form (9), we have:

$$\mid STFT(\omega,t) \mid = \mid G(\omega) \mid\mid \delta[t + \varphi'(\omega)]$$

$$*_t w(t) *_t \mathcal{F}^{-1}[e^{j\varphi''(\omega+\theta_1)\theta^2/2!}] \mid \quad (29)$$

As an example, let us consider $x(t) = \delta(t - t_1)$ with $X(\omega) = e^{-j\omega t_1}$. We see that $\mid STFT(\omega,t) = \delta(t-t_1) *_t w(t)$, i.e. the time resolution depends only on the width of $w(t)$, which is exactly what we expected. However, if the variation of phase $\varphi(\omega)$ is not linear then the additional artifacts exist.

B. Wavelet Transform

The definition of a continuous wavelet transform, of form (15), in the frequency domain is, [2]:

$$WT(a,t) = \frac{e^{jt\omega_0/a}}{2\pi\sqrt{|a|}} \int_{-\infty}^{\infty} X(\frac{\theta+\omega_0}{a})$$

$$\times W^*(\theta)e^{j\theta t/a}d\theta \quad (30)$$

For signals defined by (9) we get:

$$\mid WT(a,t) \mid = \frac{1}{\sqrt{|a|}} \mid G(\omega) \mid\mid \delta[t+\varphi'(\omega_0/a)]$$

$$*_t w(t/a) *_t \mathcal{F}^{-1}[e^{j\varphi''(\frac{\omega_0+\theta_1}{a})\theta^2/2!}] \mid \quad (31)$$

For small a the window $w(t/a)$ is narrow and the time resolution is high. At the same time if $\varphi''(\omega)$ is a constant, the artifacts do not depend on a. But if $\varphi(\omega)$ is of the higher than the second order, then the influence of a to the artifacts may be significant. An analysis of wavelet transform, in the case of asymptotic signals, may be also found in [19].

C. Generalized Wigner Distribution

The frequency domain definition of the PGWD, eqn.(18b), is:

$$PGWD(\omega,t) = \frac{1}{\mid 1-4\alpha^2 \mid} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} X(\frac{2\omega+\theta}{2+4\alpha})$$

$$\times X^*(\frac{2\omega-\theta}{2-4\alpha}) \cdot e^{j(\frac{2\omega+\theta}{2+4\alpha}-\frac{2\omega-\theta}{2-4\alpha})t}d\theta *_\omega W_s(\omega)$$

$$(32)$$

For $\alpha = 0$, i.e. the pseudo Wigner distribution, we get:

$$PWD(\omega,t) = \frac{1}{2\pi} \mid G(\omega) \mid^2 \delta[t+\varphi'(\omega)]$$

$$*_\omega W_s(\omega) *_t \mathcal{F}^{-1}[e^{j\varphi'''(\omega+\theta_1)\theta^3/12}] \quad (33)$$

We see that up to the quadratic phase variation of $\varphi(\omega)$ the artifacts do not appear. The time resolution for signals $x(t) = \delta(t-t_1)$ is ideal. If the phase $\varphi(\omega)$ is a quadratic function then, for a wide $w(t)$, the time resolution is close to the ideal, as well. Here, this requirement for a wide window is not contradictory to the high frequency resolution, if the frequency variation is linear.

If $\alpha \neq 0$ and $\alpha \neq 1/2$ it can be shown that even the position of the group delay is biased if $\varphi'(\omega)$ is not a constant.

The Rihaczek distribution, in the frequency domain, is:

$$RD(\omega,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)X^*(\omega+\theta)e^{-j\theta t}d\theta$$

$$= \mid G(\omega) \mid^2 \delta[t+\varphi'(\omega)] *_t \mathcal{F}^{-1}[e^{j\varphi''(\omega+\theta_1)\theta^2/2!}]$$

$$(34)$$

If the phase $\varphi(\omega)$ is linear then the RD and the WD produce the same resultsfor signals defined by (9).

D. Wavelet Wigner Distribution - WWD

The WWD in the frequency domain has the form:

$$PWWD(a,t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} X(\theta+\frac{\mu}{2})$$

$$\times X^*(\theta-\frac{\mu}{2})e^{j\mu t}d\mu *_\theta \mid a \mid W_s(a\theta)_{/\theta=\omega_0/a}$$

For signals defined by (9), we have:

$$PWWD(\omega,t) = \frac{1}{2\pi} \mid G(\theta) \mid^2\mid \delta[t+\varphi'(\theta)]$$

$$*_\theta|a|W_s(a\theta) *_t \mathcal{F}^{-1}[e^{j\varphi'''(\omega+\theta_1)\theta^3/12}] \mid_{/\theta=\omega_0/a}$$

$$(35)$$

Formal analysis is the same as for the PWD. If we suppose that the phase is a constant, for example $x(t) = \delta(t-t_1)$ with $\varphi'(\omega) = -t_1$, then we have $PWWD(a,t) = w(0)\delta(t-t_1)$. The time resolution is not dependent upon a. If $\varphi'(\omega) = -c\theta$ then

$$PWWD(a,t) =$$

$$\frac{1}{2\pi} \mid G(\theta) \mid^2 |\frac{a}{c}|W(a(\theta+t/c))_{/\theta=\omega_0/a}$$

$$= \frac{1}{2\pi} \mid G(\omega) \mid^2\mid \frac{a}{c} \mid W((t-c\omega)a/c).$$

We see that the concentration of energy density is around group delay. There are no artifacts, except the spread due to the window. That spread is heavily dependent on a. Here, again we have that if a increases, the time resolution is better, as in the case of the frequency resolution.

E. Scaled Wigner Distribution - LWD

The closed form analysis of group delay for the LWD can not be easily performed. But, we can conclude from eqn. (24) that if the signal $x(t)$ is time-limited (i.e. $x(t) = 0$ for $\mid t \mid > T$) then the PLWD is also time limited ($PLWD(\omega, t) = 0$ for $\mid t \mid > T$). It means that the PLWD, in the case of short signals, has the ideal time resolution, (see also [23],[29]).

V. MULTICOMPONENT SIGNALS

The previous analysis will be applied to the multicomponent signals of the form:

$$x(t) = \sum_{i=1}^{M} g_i(t) e^{j\phi_i(t)} \tag{36}$$

where $g_i(t)$ are the slow-varying amplitudes.

The STFT of the signal given by (36) is:

$$STFT(t, \omega) = \frac{1}{2\pi} \sum_{i=1}^{M} g_i(t) e^{j\phi_i(t)} \delta[\omega - \phi_i'(t)]$$
$$*_\omega W(\omega) *_\omega \mathcal{F}[e^{j\phi_i''(t+\tau_1)\tau^2/2}] \tag{37}$$

Assuming that $\phi_i"(t + \tau_1)$ is negligible inside the window, we have the spectrogram:

$$SPEC(t, \omega) = \sum_{i=1}^{M} \sum_{j=1}^{M} g_i(t) g_j^*(t) e^{j[\phi_i(t) - \phi_j(t)]}$$
$$\times W[\omega - \phi_i'(t)] W^*[\omega - \phi_j'(t)] \tag{38}$$

Suppose that the values of $W(\omega)$ may be considered as being zeros for $\mid \omega \mid \geq W_B/2$ (where W_B is the width of $W(\omega)$ or the width of its main lobe). In that case we can distinguish two cases:

1). If $\min[\mid \phi_i'(t) - \phi_j'(t) \mid] > W_B$ for all i, j and a given t, then:

$$SPEC(t, \omega) = \sum_{i=1}^{M} g_i^2(t) W^2[\omega - \phi_i'(t)] \tag{39}$$

i.e, the signal energy is concentrated in the auto terms centered at auto frequencies. Based on expression (37) we can define the instantaneous frequency of the multicomponent signal (36) as the set of M instantaneous frequencies $\{\phi_1'(t), \phi_2'(t), \ldots, \phi_M'(t)\}$.

This may be interpreted by the analogy to mechanical motion, in the following way: Signal $x(t)$ is represented by the vector describing the trajectory in the complex plane. This trajectory can be treated as the composition of M, approximately circular, trajectories with almost constant radii $g_i(t)$ and phases $\phi_i(t)$. The instantaneous frequency of each $g_i(t)$ is one of the instantaneous frequencies of the signal $x(t)$, Fig.1.

2). If, for any l and k $\mid \phi_l'(t) - \phi_k'(t) \mid < W_B$, then between the instantaneous frequencies $\phi_l'(t)$ and $\phi_k'(t)$ we have the energy of cross terms $g_l(t)e^{j\phi_l(t)}$ and $g_k(t)e^{j\phi_k(t)}$. The spectrogram, in this case, has the form:

$$SPEC(t, \omega) = \sum_{i=1}^{M} g_i^2(t) W[\omega - \phi_i'(t)]$$
$$+ g_l(t) g_k^*(t) e^{j[\phi_l(t) - \phi_k(t)]}$$
$$\times W[\omega - \phi_l'(t)] W^*[\omega - \phi_k'(t)] \tag{40}$$

Next, we will consider the PWD, which may be written using the STFT, as:

$$PWD(t, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} STFT(t, \omega + \theta)$$
$$\times STFT^*(t, \omega - \theta) d\theta \tag{41}$$

For signals whose spectrogram is of form (37), we have:

$$PWD(t, \omega) = \frac{1}{\pi} \sum_{i=1}^{M} \sum_{j=1}^{M} g_i(t) g_j^*(t) e^{j[\phi_i(t) - \phi_j(t)]}$$
$$\times \int_{-\infty}^{\infty} W[\omega + \theta - \phi_i'(t)] W^*[\omega - \theta - \phi_j'(t)] d\theta \tag{42}$$

If $W(\omega) = 0$ for $\mid \omega \mid \geq W_B/2$, then in the double summation in (40), only the terms for which:

$$\mid \omega + \theta - \phi_i'(t) \mid < W_B/2$$

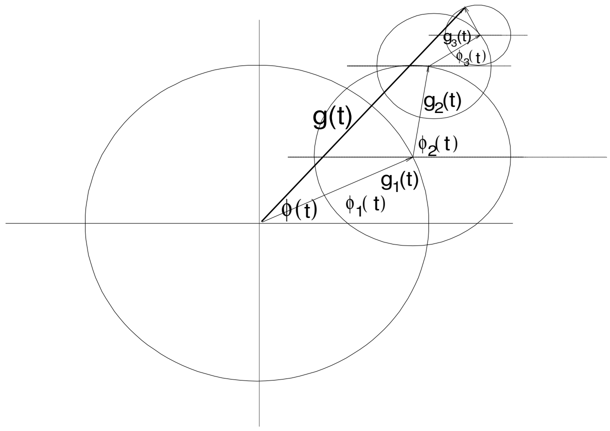

Fig. 1. Polar representation of a multicomponent signal

and

$$| \omega - \theta - \phi_j'(t) | < W_B/2 \tag{43}$$

are different from zeros.

Summing the previous inequalities, we get:

$$| \omega - \frac{\phi_i'(t) - \phi_j'(t)}{2} | < W_B/2 \tag{44}$$

The pseudo Wigner distribution exists around the frequencies $\omega = \frac{\phi_i'(t)+\phi_j'(t)}{2}$ for all i and j. It is evident that for $i \neq j$, the cross terms exist even if $\phi_i'(t)$ and $\phi_j'(t)$ are very far apart (in contrast to the spectrogram). Cross terms are centered between the i-th and j-th instantaneous frequencies. It is extremely interesting to investigate the location of the components contributing to the cross terms on the θ axis, eqn. (40). From (41) we obtain:

$$| \theta - \frac{\phi_i'(t) + \phi_j'(t)}{2} | < W_B/2 \tag{45}$$

The auto terms (for $i = j$) are obtained by the integration around $\theta = 0$ in the interval $| \theta | < W_B/2$, while the cross terms are obtained by integrating along the interval $| \theta - [\phi_i'(t) - \phi_j'(t)]/2 | < W_B/2$. This is an interesting conclusion because we can eliminate the cross terms which are apart more than W_B, and at the same time perform the complete integration over the auto terms, using the window $P(\theta)$ in integral (39), which has the width $W_P, P(\theta) = 0$ for $| \theta | > W_P/2$, such that:

$$W_B < W_P < \min_{i,j} |\phi_i'(t) - \phi_j'(t)| - W_B \tag{46}$$

In that way, we arrived at the method for time frequency analysis which will preserve the appealing properties of the Wigner distribution, but without (or with reduced) cross terms:

$$SM(t,\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} P(\theta) STFT(t, \omega + \theta) \times STFT^*(t, \omega - \theta) d\theta \tag{47}$$

This formula[6] shows very interesting effects and may be numerically more efficient than the Wigner distribution calculation itself. That is shown in [14], (see Appendix B). The cross terms suppressing was also effectively treated by the Choi-Williams method [1], with preservation of marginal properties, but in a computationally extremely intensive way. An analysis of multicomponent signals and cross term effects may be found in [24], as well.

Similarly, the L-Wigner distribution, (24), can be understood as a convolution of the pseudo Wigner distributions. For $L = 2$ we get:

$$LWD(t,\omega) = \frac{1}{\pi} PWD(t, 2\omega) *_\omega PWD(t, 2\omega) \tag{48}$$

[6]Two special cases of $SM(t,\omega)$ are: 1) If $P(\theta) = \pi\delta(\theta)$, then $SM(t,\omega) = SPEC(t,\omega)$;2) If $P(\theta) \equiv 1$, then $SM(t,\omega) = PWD(t,\omega)$.

The modified pseudo L-Wigner distribution (MPLWD) for cross terms elimination is:

$$MPLWD(t,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} P_L(\theta)PWD(t,\omega+\theta)$$
$$\times PWD(t,\omega-\theta)d\theta \qquad (49)$$

where the properties of the window $P_L(\theta)$ are the same as for $P(\theta)$ in eqn. (44). Convolving, further, two L-Wigner distributions for $L=2$, we get the L-Wigner distribution for $L=4$, and so on. That way we can achieve the energy concentration described in (25), at the same time avoiding the cross terms.

Let us consider the multicomponent signal of the form:

$$X(\omega) = \sum_{i=1}^{M} G_i(\omega)e^{j\varphi_i(\omega)} \qquad (50)$$

As in (36), we get:

$$SPEC(t,\omega) = \sum_{i=1}^{M}\sum_{j=1}^{M} G_i(t)G_j^*(t)$$
$$\times e^{j[\varphi_i(\omega)-\varphi_j(\omega)]}\cdot w[t+\varphi_i'(\omega)]w^*[t+\varphi_j'(\omega)] \qquad (51)$$

If the window $w(t)$ width is T_w, i.e. $w(t)=0$ for $|\ t\ |\geq T_w/2$, then we can consider two cases. First, when the distance between each $\varphi_i'(\omega)$ and $\varphi_j'(\omega)$ is at least T_w, then only auto terms exist. In the second case, when some $\varphi_i'(\omega)$ and $\varphi_j'(\omega)$ are closer than T_w, the cross terms exist.

Defining the pseudo Wigner distribution in the frequency domain by:

$$PWD_f(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega+\theta/2)$$
$$\times X^*(\omega-\theta/2)\cdot W(\theta/2)W^*(-\theta/2)e^{j\theta t}d\theta \qquad (52)$$

and using the analogy with the case of instantaneous frequency, eqns. (36)-(44), we may define the method for elimination of cross terms along the time axis. Taking into consideration that:

$$PWD_f(t,\omega) = \int_{-\infty}^{\infty} STFT(t+\tau/2,\omega)$$
$$\times STFT^*(t-\tau/2,\omega)e^{-j\omega\tau}d\tau \qquad (53)$$

and neglecting the artifacts, we get:

$$PWDL_f(t,\omega) = \sum_{i=1}^{M}\sum_{j=1}^{M} G_i(\omega)G_j^*(\omega)$$
$$\times e^{j[\varphi_i(\omega)-\varphi_j(\omega)]}\int_{-\infty}^{\infty} w[t+\tau/2+\varphi_i'(\omega)]$$
$$\times w^*[t-\tau/2+\varphi_j'(\omega)]d\tau \qquad (54)$$

The auto terms and cross terms analysis can be done as in the case of the instantaneous frequency. Cross terms are concentrated around $\tau=\varphi_i'(\omega)-\varphi_j'(\omega)$. Using the window $p(\tau)$ they can be eliminated. On the basis of the above analysis we can define the distribution in the form:

$$SM_f(t,\omega) = \int_{-\infty}^{\infty} p(\tau)STFT(t+\tau/2,\omega)$$
$$\times STFT^*(t-\tau/2,\omega)e^{-j\omega\tau}d\tau \qquad (55)$$

which will, under the described conditions, have only auto terms. The characteristics of the window $p(\tau)$, as well as the window $p(\tau)$ itself, may be taken into acount selecting $w_s(\tau)$.

For the multicomponent signals of form (36) and wavelet transforms, the scalogram is in the form:

$$SCAL(t,\omega) = \sum_{i=1}^{M}\sum_{j=1}^{M} |\ a\ |\ g_i(t)g_j(t)$$
$$\times e^{j[\phi_i(t)-\phi_j(t)]}\cdot W[\omega_0-a\phi_i'(t)]W[\omega_0-a\phi_j'(t)] \qquad (56)$$

where the artifacts are neglected. Further analysis may be easily pursued following the spectrogram case.

The pseudo WWD can be written in form of the WT's convolution:

$$PWWD(a,t) = \frac{1}{\pi}WT(a,t,\theta) *_\theta WT(a,t,\theta)$$
$$\text{at } \theta = 2\omega_0,$$

where

$$WT(a,t,\theta) = \sqrt{|a|}\int_{-\infty}^{\infty} x(t+a\tau)w^*(\tau)e^{-j\theta\tau}d\tau,$$

and $WT(a,t,\omega_0)=WT(a,t)$.

For signals whose scalogram is given by (53) we have:

$$WWD(t,\omega)=\frac{1}{\pi}\sum_{i=1}^{M}\sum_{j=1}^{M}\mid a\mid g_i(t)g_j^*(t)$$
$$\times e^{j[\phi_i(t)-\phi_j(t)]}\int_{-\infty}^{\infty}W[\omega_0+\theta-a\phi_i'(t)]$$
$$\times W^*[\omega_0-\theta-a\phi_j'(t)]d\theta \qquad (57)$$

The SM form of the PWWD is:

$$SMW(\omega,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}P(\theta)WT(a,t,\omega_0+\theta)$$
$$\times WT^*(a,t,\omega_0-\theta)d\theta \qquad (58)$$

If $P(\theta)=\pi\delta(\theta)$, then the SMW is equal to the scalogram. Using $P(\theta)$ with an appropriate width, we can keep the properties of the PWWD, avoiding the expected cross terms.

For computational realization, the SMW can be understood as a convolution of the WT's that are calculated using different basis wavelet functions $h_i(t)=w(t)e^{j\omega_i t}$, having frequencies ω_i in the vicinity of and at ω_0. The window $P(\theta)$ may be dependent on a.. Namely, if a is a small number, then the artifacts are reduced in the WT itself, eqn. (17), thus removing the need for the convolution and additional WT calculations.

The analysis of the group delay in the wavelet transforms is formally identical with the one in the case of STFT.

VI. ANALYSIS OF THE ALIASING EFFECTS

The aliasing effects are interesting in the WD, [15], [16]. Because of the quadratic nature of the WD, the signal has to be oversampled by factor 2 with respect to the sampling interval defined in sampling theorem. Another way to avoid aliasing in the WD is by use of an analytic signal [4],[5]. Here we will show that the aliasing components, appearing in the WD, may be eliminated in the same way as the cross terms.

Consider the discrete signal $x_d(t)$, obtained by sampling a continuous signal $x(t)$.

$$x_d(t)=\sum_{n=-\infty}^{\infty}Tx(nT)\delta(t-nT) \qquad (59)$$

where T represents the sampling interval.

The Fourier transform of $x_d(t)$ is a periodic function along frequency axis, with period $\omega_p=2\pi/T$, and has the form [7]:

$$X_d(\omega)=\sum_{k=-\infty}^{\infty}X[(\omega+k\omega_p] \qquad (60)$$

We see that the formal analysis is the same as for multicomponent signals.

The STFT for the sampled signal (56) is of the form:

$$STFT_d(n,\omega)=g(t)e^{j\phi(t)}$$
$$\times\sum_{k=-\infty}^{\infty}W[(\omega+k\omega_p-\phi'(t)]_{|t=nT} \qquad (61)$$

where we have neglected the distortions due to higher-order derivatives of the phase function.

Combining the previous relation with (37), the Wigner distribution of discretized signal is obtained:

$$WD_d(n,\omega)=\frac{1}{\pi}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\mid g(t)\mid^2$$
$$\times\int W[\omega+\theta+k_1\omega_p-\phi'(t)]$$
$$\times W^*[\omega-\theta+k_2\omega_p-\phi'(t)]_{|t=nT}d\theta \qquad (62)$$

Using similar procedure as in the cross terms analysis, it may be seen that the integrand in (59) is nonzero if the following holds:

$$-W_B/2-(k_1-k_2)\frac{\omega_p}{2}<\theta<W_B/2-(k_1-k_2)\frac{\omega_p}{2} \qquad (63)$$

It is obvious that auto terms ($k_1=k_2$) appear as a consequence of integration around the origin in the θ coordinate system. The closest aliasing components along θ axis are those for $k_1-k_2=\pm1$. Obviously, they may be eliminated by using a window $P(\theta)$, which is equal to zero along the θ axis, for the values of θ outside the interval $\mid\theta\mid<\omega_p/2-W_B/2$. Observe that this condition is usually significantly relaxed, as compared to the condition for eliminating cross terms, eqn. (45). Thus, removal of cross terms by the SM usually guarantees the elimination of the aliasing components.

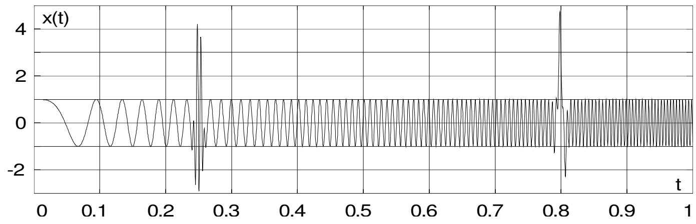

Fig. 2. Signal $x(t)$

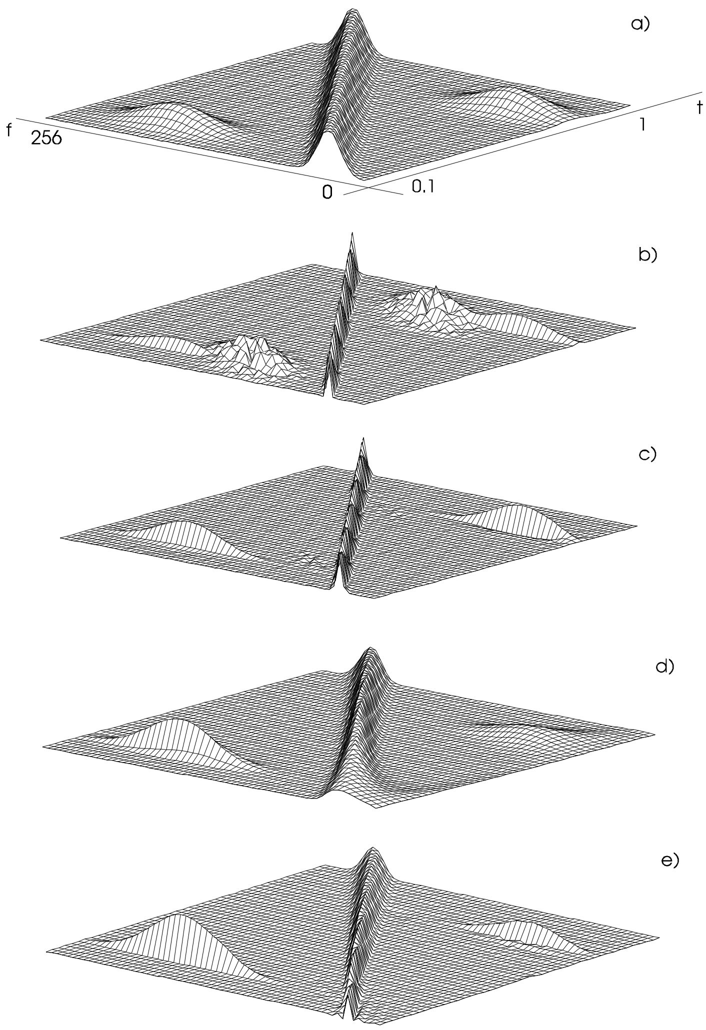

Fig. 3. Time-frequency representation of the linear frequency modulated signal with two Gaussian chirp pulses: a) Spectrogram; b) Pseudo Wigner Distribution; c) Modified pseudo Wigner distribution - SM; *d*) Scalogram; *e*) Modified pseudo Wavelet Wigner distribution -SMW; with: The Hanning window $w(t)$ whose width is $T = 0.25$; The rectangular window $P(\theta)$ of the width $W = 128\pi$; In the scalogram and in the SMW $\omega_0 = 64\pi$.

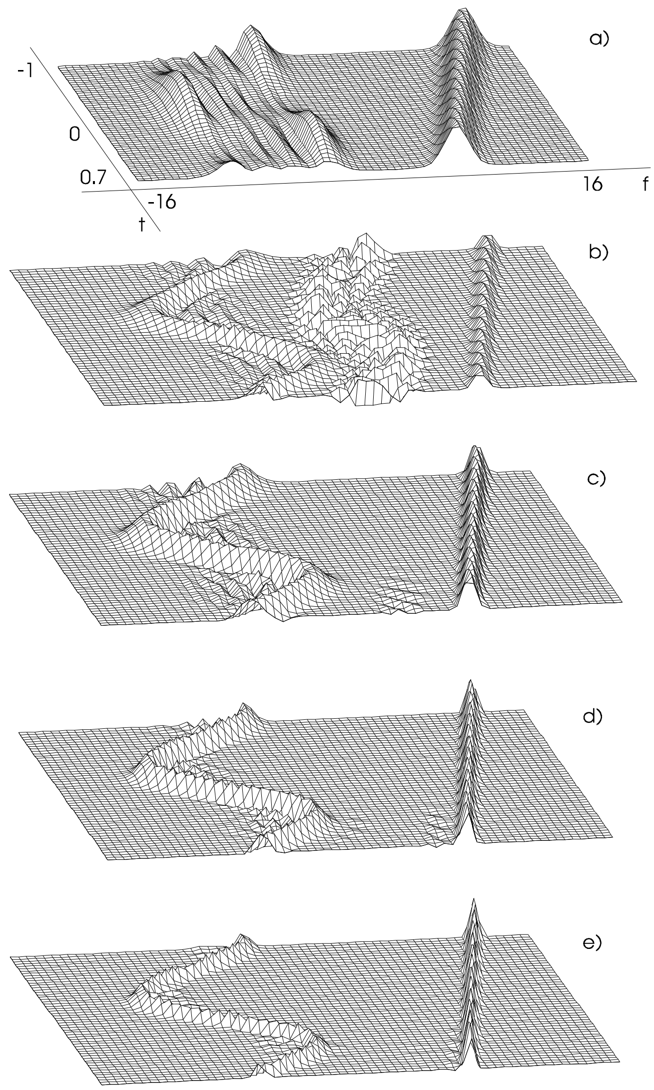

Fig. 4. Time-frequency representation of linear and sinusoidal frequency modulated signals: a) Spectrogram, *b*) Pseudo Wigner Distribution, *c*) Modified pseudo Wigner distribution - SM; *d*) Modified L-Wigner distribution (MPLWD) with $L = 2$; *e*) Modified L-Wigner distribution with $L = 4$; With: The Hanning window $w(t)$ whose width is $T = 2$; Window $P(\theta)$ is rectangular whose width is $W = 6.5\pi (N = 64, Ld = 3)$.

The complete above analysis may be extended to the case of multidimensional signals. Some results dealing with the multidimensional case are reported in [27].

VII. NUMERICAL EXAMPLE

Let us consider a signal $x(t)$, that is the sum of a linearly frequency modulated signal and two Gaussian chirp pulses:

$$x(t) = A_1 e^{j\beta_1 t^2} + A_2 e^{-\alpha_2 (t-t_{12})^2} e^{j\beta_2 t^2} + A_3 e^{-\alpha_3 (t-t_{13})^2} e^{j\beta_3 t^2} \quad (64)$$

with: $A_1 = 1, A_2 = A_3 = 4, \alpha_2 = \alpha_3 = 150, \beta_1 = 700, \beta_2 = 190, \beta_3 = 2225, t_{12} = 0.8, t_{13} = 0.25$.

The first component of $x(t)$ is of the form described by eqn.(4), while the second and third ones are of form (9), Fig.2.

The spectrogram of $x(t)$ is shown in Fig. 3a. The concentration of energy is improved by the PWD, Fig 3b. But, because of the quadratic nature of the WD, the cross terms are very emphatic. The cross terms in the WD can be decreased or completely removed if the signal components do not overlap in the time-frequency plane using the SM Fig. 3c.

In Fig. 3d, the scalogram of $x(t)$ is shown. The modified version of the PWWD (SMW) is given on Fig 3e.

As a second example, we consider a sum of the sinusoidal frequency modulated signal and linearly frequency modulated signal:

$$x(t) = e^{-j3.5\pi(t-2.5)^2} + e^{j\{6\sin[1.5\pi(t+1)]-12\pi t\}} \tag{65}$$

We calculated the STFT (Fig. 4a) and the PWD (Fig. 4b) using a Hanning window, as well as the SM (Fig. 4c) and modified pseudo L-Wigner distribution (for $L = 2$ and $L = 4$, Figs. 4d and 4e) with the same number of samples. The improvement of energy density concentration around the instantaneous frequency, as well as the cross terms reduction (removal) using the modified pseudo L-Wigner distributions is clearly shown in Fig.4. For the details on the numerical implementation see Appendix B.

VIII. CONCLUSION

The comparison of commonly used energy time-frequency distributions with the distributions having the ideal instantaneous frequency or group delay representation is accomplished. It is shown that the WD is the best among them. A wavelet and scaled definition of the WD are introduced. The analysis of cross terms in the case of the multicomponent signals is performed and the method for their removal is presented. The results are demonstrated on the numerical examples with the frequency modulated signals.

IX. ACKNOWLEDGMENT

The author is grateful to the anonymous reviewer for his helpful comments, as well as for bringing into author's attention same very useful references.

X. APPENDIX A

A Simple Derivation of the Cohen Class of Distriburions

Let $P_x(t,\omega)$ be an arbitrary two-dimensional function. Its inverse Fourier transform will be denoted by $M(\theta,\tau)$. The Fourier transform pair, taking into account physical properties of the particular variables, is given by:

$$M(\tau,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_x(\omega,t)e^{j\theta t+j\omega\tau}d\omega dt$$

$$P_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(\theta,\tau)e^{-j\theta t-j\omega\tau}d\theta d\tau \tag{66}$$

The property described by eqn. (1) is satisfied if $M(0,0) = E_x$, and the marginal properties (2) and (3) are fulfilled if and only if:

$$M(\theta,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} P_x(\omega,t)d\omega]e^{j\theta t}dt$$

$$= \int_{-\infty}^{\infty} \mid x(t) \mid^2 e^{j\theta t}dt \tag{67}$$

$$M(0,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} P_x(\omega,t)dt]e^{j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mid X(\omega) \mid^2 e^{j\omega\tau}d\omega \tag{68}$$

From the previous relations and the uniqueness of the one-dimensional and two-dimensional Fourier transform we can make the following conclusions:

1^o The general form of a time-frequency distribution $P_x(t,\omega)$ is:

$$P_x(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(\theta,\tau)e^{-j\theta t-j\omega\tau}d\theta d\tau \tag{69}$$

where $M(\theta,\tau)$ is an arbitrary function for all (θ,τ) except on the lines $\theta = 0$ and $\tau = 0$ where it has to have the values defined by (A2) and (A3) in order to satisfy the marginals (2) and (3). If only the unbiased energy condition is sufficient, then $M(\theta,\tau)$ is an arbitrary function elsewhere except at the point (0,0) where $M(0,0) = E_x$.

2^o If we know only one distribution $P_x(t,\omega)$ with the inverse two-dimensional Fourier transform $M(\theta,\tau)$, satisfying the marginal properties, then a general function $P_{xg}(t,\omega)$ having inverse Fourier transform $M(\theta,\tau)$ multiplied by an arbitrary function $c(\theta,\tau)$:

$$M_g(\theta,\tau) = M(\theta,\tau)c(\theta,\tau) \tag{70}$$

satisfied marginals as well, if $c(\theta,0) = c(0,\tau) = 1$. This way one may construct an infinite number of time-frequency distributions, knowing just one of them. The general distribution, combining (70),(69) and (66), is:

$$P_{xg}(t,\omega) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\theta,\tau)$$

$$\times P_x(s,u)e^{j\theta s + ju\tau - j\theta t - j\omega\tau}dudsd\theta d\tau \tag{71}$$

Note: Taking the Wigner distribution, (19), as a basis $P_x(s,u)$ in (71) the Cohen class of distributions (20) is obtained.

4^o The general distribution $P_{xg}(t,\omega)$ preserves very important properties of time and frequency shifts. If the particular distribution $P_x(t,\omega)$ in (71) is time and frequency shifted, i.e. the basis is $P_{xs}(t,\omega) = P_x(t-t_0,\omega-\omega_0)$, then the general distribution is time and frequency shifted as well, $P_{xgs}(t,\omega) = P_{xg}(t-t_0,\omega-\omega_0)$. This directly follows from (71).

XI. APPENDIX B

Discrete Signals

The discrete form of the spectrogram, whose STFT is given by (12), is:

$$DSPEC(n,k) = \mid DSTFT(n,k) \mid^2$$

$$= \mid \sum_{i=-N/2+1}^{N/2} w(i)x(n+i)\bar{e}^{j\frac{2\pi}{N}ik} \mid^2$$

$$= \mid \sum_{i=0}^{N-1} x_n(i)W_N^{ik} \mid^2 \tag{72}$$

The meaning of $x_n(i)$ and W_N in (72) is obvious.

The discrete pseudo WD in the time domain (eq.(20) with $\alpha = 0$) is:

$$DPWD(n,k) = 2\sum_{m=-N+1}^{N} w(m)w(-m)$$

$$\times x(n+m)x^*(n-m)W_{2N}^{2mk} \tag{73}$$

where the discrete signal and window in (73) are sampled with 1/2 of the sampling interval assumed in (72).

The discretization of the SM (eqn.(39)) produces:

$$DSM(n,k) = \sum_{i=-Ld}^{Ld} Pd(i)$$

$$\times DSTFT(n,k+i)DSTFT^*(n,k-i) \tag{74}$$

where $2Ld+1$ is the width of discrete window $Pd(i)$. We see that if $Pd(i) = \delta(i)$ then $DSM(n,k) = DSPEC(n,k)$. Noting that:

$$DSTFT(n,k+i)DSTFT^*(n,k-i)$$

$$+DSTFT(n,k-i)DSTFT^*(n,k+i)$$

$$= 2Real\{DSTFT(n,k+i)DSTFT^*(n,k-i)\} \tag{75}$$

and assuming $Pd(i)$ is a rectangular window, we have:

$$DSM(n,k) = DSPEC(n,k) +$$

$$2\sum_{i=1}^{Ld} Real\{DSTFT(n,k+i)$$

$$\times DSTFT^*(n,k-i)\} \tag{76}$$

For the WD calculation the sampling interval has to be less than one half of the sampling interval specified by the sampling theorem, [17]. In the frequency domain this means that the calculation of the convolution can be performed using the FFT after an appropriate zero padding. For the DSM zero padding in the frequency domain (i.e. oversampling in time domain) is not necessary, because the aliasing components will be removed in the same way as the cross terms. If we assume that $DSTFT(n,k) = DSTFT(n,k+N)$, the aliasing may occur only in the marginal intervals whose width is equal to the width of $Pd(i)$. But, this is not a necessary assumption, because eqn. (76) allow a direct calculation. The terms containing the values of $DSTFT(n,k\pm i)$ outside the basic period, can simply be omitted. The worst case, the last marginal values of the DSM, will be always equal to the values of the spectrogram at these points.

The alternative, commonly used, way to avoid oversampling for the WD is in using the analytic signal.

To calculate the DSM we need to calculate the Fourier Transform at the time instant n. This can be done by a recursive formula, from the previous values of the Fourier Transform at the time instant $n-1$, [7]:

$$DSTFT(n+1,k) = [x_n(N) - x_n(0) + DSTFT(n,k)]e^{j\frac{2\pi}{N}k}$$

or substituting $x_n(N)$ and $x_n(0)$, we get:

$$DSTFT(n+1,k) = [x(n+N/2+1) - x(n-N/2+1)](-1)^k + DSTFT(n,k)e^{j\frac{2\pi}{N}k} \tag{77}$$

The initial Fourier Transform calculation has to be performed using the FFT routine. All the subsequent calculations can be done recursively according to eqn. (77).

Equation (77) gives the Fourier coefficients when the rectangular window $w(n)$ is applied. If we use, for example, the Hanning window, then the coefficients should be modified by:

$$DSTFT_H(n,k) = \frac{1}{2}\{DSTFT(n,k) + \frac{1}{2}[DSTFT(n,k-1) + DSTFT(n,k+1)]\} \tag{78}$$

Accordind to eqn.(46) we may calculate $2L$-Wigner distribution from L-Wigner distribution:

$$DMP_{2L}WD(n,k) = DMP_L WD^2(n,k) + 2\sum_{i=1}^{Ld} DMP_L WD(n,k+i) \times DMP_L WD(n,k-i)\}$$

This way, the resulting L-Wigner distribution is cross-terms free if the STFT is cross-terms free (what is the case if the signal components do not overlap in the time-frequency plane) and if, at the same time, the conditions for cross-terms elimination $(43a)$ is satisfied in each iteration. Note, if $(43a)$ can not be satisfied for some i, j and t then the cross terms will appear at the instant t, between the i-th and j-th signal component.

The details of the numerical realization described in this appendix may be found in [14,29].

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