An Analysis of Instantaneous Frequency Representation Using Time-Frequency Distribution-Generalized Wigner Distribution

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Abstract— **This paper presents an analysis of the representation of instantaneous frequency using time-frequency distributions of energy density domain. Similarity to the "ideal" instantaneous frequency presentation is chosen as a criterion for comparison of various distributions. Although all the commonly used distributions suffer from the artifacts along frequency axis, it is shown that the Wigner distribution is the best among them, with respect to this criterion. The generalization of Wigner distribution - LWD is introduced to decrease the artifacts. The properties of the LWD are analyzed. It is shown that, at the expense of an insignificant increase in computation time, much better results are obtained. The theory is illustrated by a numerical example with the frequency modulated signals.**

I. INTRODUCTION

Time-frequency distributions have been intensively studied during the last decade. We refer to several excellent review papers on distributions for time-frequency analysis [1,2,3]. The commonly used distributions of energy density domain (E-domain, [5]) are the following: Short time Fourier transform - STFT (via spectrogram), Wigner distribution -WD and their variations - generalized Wigner distribution - GWD, [4,5]. All of these distributions exhibit significant artifacts along frequency axis. It is known, however, that only the WD in the case of linear frequency modulated signal, does not suffer from the artifacts [9].

The paper is organized as follows. In Section II we analyze the artifacts along the frequency axis. This analysis is used in Section III to a generalization of the WD (L-Wigner distribution) with the appealing properties. Namely, it retains the computational simplicity, while preserving important properties of the WD, and reducing the artifacts. Section IV contains numerical example.

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II. ANALYSIS OF INSTANTANEOUS FREQUENCY PRESENTATION

A criterion for comparison of time-frequency distributions may be defined in various ways [8,9]. Here, we will assume that the "ideal" time frequency distribution is the one producing the Dirac pulse at the instantaneous frequency of an arbitrary frequency modulated signal; elsewhere the value of the distribution should be zero.

Complex signals will be considered. If a signal is real, its analytic part will be taken. A complex signal can be represented in the form:

$$f(t) = r(t)e^{j\phi(t)} \tag{1}$$

We will first consider a signal $f(t)$ with no amplitude variation:

$$f(t) = e^{j\phi(t)} \tag{2}$$

which is of practical interest in many areas.

The ideal time frequency transform (ITFT) should produce the Dirac pulse at the instantaneous frequency $\omega_i(t) = \phi'(t)$. It should be in the form:

$$ITFT(\omega, t) = 2\pi\delta(\omega - \phi'(t)). \tag{3}$$

If a window is used, we will consider $IWTFT(\omega,t) = ITFT(\omega,t) * W(\omega) = W(\omega -$

$\phi'(t))$, where $W(\omega)$ is the Fourier transform (FT) of the window $w(\tau)$.

The widely used time-frequency distributions of E-domain will be compared with respect to the defined criterion.

The oldest way of the time-frequency signal representation is based on the use of the short time Fourier transform (STFT), [3,6,7]:

$$STFT(\omega,t)=\int_{-\infty}^{\infty} f_w(t+\tau)e^{-j\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} w(\tau)f(t+\tau)e^{-j\omega\tau}d\tau \qquad (4)$$

where $f_w(t+\tau)$ is a signal truncated by the window $w(\tau)$, i.e. $f_w(t+\tau)=f(t+\tau)w(\tau)$.

Expanding $\phi(t+\tau)$, into a Taylor series, around t, we get:

$$STFT(\omega,t)=\int_{-\infty}^{\infty} w(\tau)$$

$$\times e^{j[\phi(t)+\phi'(t)\tau+\phi"(t)\tau^2/2+\ldots]}e^{-j\omega\tau}d\tau$$

$$STFT(\omega,t)=\frac{1}{2\pi}f(t)\delta(\omega-\phi'(t))$$

$$*FT\left\{e^{j\sum_{n=2}^{\infty}\phi^{(n)}(t)\tau^n/n!}\right\}*W(\omega) \qquad (5)$$

where $*$ denotes a convolution in ω.

It is known that:

$$FT\left\{e^{j\sum_{n=2}^{\infty}\phi^{(n)}(t)\tau^n/n!}\right\}$$

$$=FT\left\{e^{j\phi^{(2)}(t+t_1)\tau^2/2!}\right\},\ 0\le t_1\le\tau.$$

From eq. (5) we see that the STFT may be interpreted as the IWTFT multiplied by $f(t)$ and convolved with the higher order terms of the derivatives of $\phi(t)$, starting with the second order term. The dominant artifacts are due to the second order term, thus being rather significant. The same conclusion is valid for the spectrograms obtained from the $STFT(\omega,t)$:

$$SPEC(\omega,t)=|STFT(\omega,t)|^2$$

$$=\left|\frac{1}{2\pi}W(\omega-\phi'(t))*FT\left\{e^{j\phi^{(2)}(t+t_1)\tau^2/2!}\right\}\right|^2.$$

A very often used distribution of E-domain is the Wigner distribution (WD), whose generalized form (GWD) is:

$$GWD(\omega,t)=\int_{-\infty}^{\infty} f(t+(\alpha+1/2)\tau)$$

$$\times f^*(t+(\alpha-1/2)\tau)e^{-j\omega\tau}d\tau \qquad (6)$$

where α is a constant.

For $\alpha=1/2$, the Rihaczek distribution (RD) or the pseudo Rihaczek distribution (PRD) are obtained, [5]:

$$RD(\omega,t)=\int_{-\infty}^{\infty} f(t+\tau)f^*(t)e^{-j\omega\tau}d\tau$$

$$PRD(\omega,t)=\int_{-\infty}^{\infty} w(\tau)f(t+\tau)f^*(t)e^{-j\omega\tau}d\tau. \qquad (7)$$

For $\alpha=0$ eq.(6) reduces to the Wigner distribution or the pseudo Wigner distribution (PWD):

$$WD(\omega,t)=\int_{-\infty}^{\infty} f(t+\frac{\tau}{2})f^*(t-\frac{\tau}{2})e^{-j\omega\tau}d\tau$$

$$PWD(\omega,t)=\int_{-\infty}^{\infty} w(\frac{\tau}{2})w(-\frac{\tau}{2})$$

$$\times f(t+\frac{\tau}{2})f^*(t-\frac{\tau}{2})e^{-j\omega\tau}d\tau. \qquad (8)$$

Expanding $\phi(t+(\alpha+1/2)\tau)$ and $\phi(t+(\alpha-1/2)\tau)$ into a Taylor series around t we get:

$$GWD(\omega,t)=\frac{1}{2\pi}\delta(\omega-\phi'(t))$$

$$*FT\left\{e^{j\sum_{n=2}^{\infty}\phi^{(n)}(t)\tau K_n/n!}\right\}*W_1(\omega) \qquad (9)$$

where $K_n=\sum_{i=0}^{n-1}[(\alpha+1/2)\tau]^{n-1-i}[(\alpha-1/2)\tau]^i$, and $w_1(\tau)$ is the equivalent window.

It is apparent that $GWD(\omega,t)$ is a convolution of the IWTFT and the FT of a function of the higher order derivatives of $\phi(t)$. A very interesting case is for the value $\alpha=0$ (WD), for which the second order term is zero, as well as the all even order terms. It means that the lowest disturbing term is of the third order. In other words, the WD is closer to the IWTFT than any other GWD for $\alpha\neq 0$.

For the WD, eq. (9) can be written in the form:

$$WD(\omega,t)=\frac{1}{2\pi}\delta(\omega-\phi'(t))$$

$$*FT\left\{e^{j\frac{\phi^{(3)}(t+t_1)+\phi^{(3)}(t-t_2)}{3!2^3}\tau^3}\right\}*W_1(\omega). \quad (10)$$

If a signal is linear frequency modulated, i.e. $\phi(t)=at+bt^2$, then the WD is identical to the IWTFT [9,23].

If the amplitude $r(t)$ is not constant, the instantaneous frequency is by definition:

$$\omega_i(t)=\frac{Im[f'(t)f^*(t)]}{|f(t)|^2}=\phi'(t). \quad (11)$$

The above analysis and conclusions are valid for the case when the amplitude $r(t+\tau)$ may be treated as a constant inside the window $w(\tau)$, i.e., when $r(t+(\alpha+1/2)\tau)r(t+(\alpha-1/2)\tau)w_1(\tau)\cong r^2(t)w_1(\tau)$.

III. GENERALIZED WIGNER DISTRIBUTION

Our intention is to define a distribution which is closer to the ITFT than the WD. Now, we will introduce one such distribution, which reduces the higher order terms, as:

$$LWD(\omega,t)=\int_{-\infty}^{\infty}f^L(t+\frac{\tau}{2L})$$

$$\times f^{*L}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau$$

$$PLWD(\omega,t)=\int_{-\infty}^{\infty}w_L(\tau)f^L(t+\frac{\tau}{2L})$$

$$\times f^{*L}(t-\frac{\tau}{2L})e^{-j\omega\tau}d\tau \quad (12)$$

where L is any integer greater than 0. We will call it the L-Wigner distribution (LWD).

Making the change of variables $s=\tau/L$, the LWD can be written in the form $LWD(\omega,t)=LWD_{f^L}(L\omega,t)$; it may simply be interpreted as a scaled WD of a signal raised to the L-th power.

This distribution locally linearizes the instantaneous frequency inside the window $w_L(\tau)$, keeping its value unchanged. This may be easily proved by expanding the phase characteristics of $f^L(t+\tau/2L)f^{*L}(t-\tau/2L)$ and $f(t+\tau/2)f^*(t-\tau/2)$ into a Taylor series:

$$L\phi(t+\tau/2L)-L\phi(t-\tau/2L)$$

$$=\tau\phi'(t)+\frac{1}{L^2}\frac{2\phi^{(3)}(t)}{3!}\left(\frac{\tau}{2}\right)^3+...;$$

$$\phi(t+\tau/2)-\phi(t-\tau/2)$$

$$=\tau\phi'(t)+\frac{2\phi^{(3)}(t)}{3!}\left(\frac{\tau}{2}\right)^3+...$$

The term $\tau\phi'(t)$, producing factor $\delta(\omega-\phi'(t))$ in the distributions, is the same for any L, but the higher terms are drastically reduced in the L-Wigner distribution (for $L>1$). The PLWD is of the form:

$$PLWD(\omega,t)=\frac{1}{2\pi}\delta(\omega-\phi'(t))$$

$$*FT\left\{e^{j2\sum_{n=3,5,...}^{\infty}\phi^{(n)}(t)\frac{\tau^n}{2^nL^{n-1}n!}}\right\}*W_1(\omega). \quad (13)$$

This form has terms of the third and higher order (the odd ones), which are divided by the factor L^{n-1}. For example, for $L=2$, the dominant third term is divided by 4 (which is equivalent to $12dB$). This gives a significant improvement over the WD.

For the direct computation of the PLWD according to (12), we have to calculate the Lth power of a signal. Signal has to be oversampled by the factor L, as compared to the WD, as well[1]. But this does not affect the computation time, since, as it will be shown, the LWD exhibits its advantages over the WD taking the same number of samples around t.

The number of additions and multiplications for the calculation of the Wigner distribution, for a single value of t, is of order $Nlog_2N$ (additions) and $N+Nlog_2N$ (multiplications),

[1]Note that a scaled signal $f(kt)$ can be obtained from the signal $f(t)$ using two quadratic filters and two quadratic modulators [6]:

$$f(kt)=A(\{[f(t)*e^{jt^2/2\beta_1}]e^{j\alpha_1t^2/2}\}$$

$$*e^{-jt^2/2\beta_2})e^{-j\alpha t^2/2}$$

where $\beta_1=(k^2-k)/\alpha,\alpha_1=\alpha/k,\beta_2=(1-k)/\alpha$ (α is an arbitrary constant and A is the constant depending on k and α).

where N is the number of samples. If we use LWD the number of multiplication is of order $Nlog_2 2L + Nlog_2 N$. For $N = 256$ and $L = 2$ the ratio of multiplication numbers for the LWD and the WD is 1.11. Having in mind that the number of additions is the same, we conclude that the total time increase is not significant.

Recently, we have derived a new method for time-frequency analysis [18]. This method may be efficiently applied in the PLWD realization, without need for oversampling, avoiding the crossterms in the case of multicomponent signals (including additive noise, [10,20]). This realization of the L-Wigner distribution is described in [21,23,24].

In the Wigner and L-Wigner distributions the lowest disturbing term is a function of the third derivative of the phase. It is possible to define a distribution with the lowest disturbing term being the function of the fifth, or even higher order derivatives. If we define a distribution in the form:

$$WD5(\omega,t) = \int_{-\infty}^{\infty} f_w^a(t+\tau/2) f_w^{*a}(t-\tau/2) \times f_w^b(t-\tau) f_w^{*b}(t+\tau) e^{-j\omega\tau} d\tau \qquad (14)$$

for signals defined by (2), we get:

$$WD5(\omega,t) = = \int_{-\infty}^{\infty} e^{j[\phi'(t)(a-2b)\tau + \phi^{(3)}(t)(a/4-2b)\tau^3/3! + \ldots} \times e^{-j\omega\tau} d\tau * W_1(\omega).$$

Taking $a - 2b = 1$ and $a/4 - 2b = 0$ ($a = 4/3$, $b = 1/6$) we get the distribution concentrated at the instantaneous frequency $\omega = \phi'(t)$ with the lowest disturbing term being of the fifth order. But, we will not further pursue this analysis, because presently it is not computationally attractive.

A. The Properties of the LWD

The following properties, equivalent to the ones of the WD [1,3], may be easily derived:

1^o The L-Wigner distribution is always real.

2^o If the signal is time shifted $f(t-t_o)$ then its LWD is time shifted as well, $LWD(\omega, t-t_o)$.

3^o The LWD of a modulated signal $f(t)e^{j\omega_o t}$ is shifted in frequency $LWD(\omega-\omega_o, t)$.

4^o Is the signal $f(t)$ is time limited, i.e. $f(t) = 0$ for $\mid t \mid > T$, then the L-Wigner distribution is time limited, $LWD(\omega,t) = 0$ for $\mid t \mid > T$.

5^o If the signal $f(t)$ is band limited with ω_m ($F(\omega) = 0$ for $|\omega| > \omega_m$), then $LWD(\omega,t)$ is limited in the frequency domain by ω_m, as well.

6^o Integral of the L-Wigner distribution over frequency is equal to the generalized signal power:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} LWD(\omega,t) d\omega = \mid f(t) \mid^{2L}.$$

7^o Integral of $LWD(\omega,t)$ over time and frequency is equal to the $2L-th$ power of the $2L-th$ norm of signal $f(t)$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} LWD(\omega,t) d\omega dt = \int_{-\infty}^{\infty} \mid f(t) \mid^{2L} dt = \|f(t)\|_{2L}^{2L}.$$

8^o The integral over time is:

$$\int_{-\infty}^{\infty} LWD(\omega,t) dt = \mid F_L(\omega) \mid^2 = \mid \underbrace{F(L\omega) * \ldots * F(L\omega)}_{\text{L-1 times}} \mid^2.$$

9^o For a large value of L ($L \to \infty$) we may neglect all values of $LWD(\omega,t)$, comparing them to the one at the point(s) (ω_m, t_m), where the distribution reaches its essential supremum[2]:

$$\lim_{L\to\infty} \frac{LWD(\omega,t)}{LWD(\omega_m,t_m)}$$

[2]Robust control widely exploits the fact that an infinity norm of a signal $f(t)$ results in its essential supremum:

$$\|f(t)\|_\infty = \lim_{p\to\infty}\left(\int_{t_1}^{t_2} |f(t)|^p dt\right)^{1/p} = = ess \sup_{t\in[t_1,t_2]} |f(t)|$$

$$= \begin{cases} 0 & if \quad \omega \neq \omega_m \ \ or \ \ t \neq t_m \\ 1 & if \quad \omega = \omega_m \ \ and \ \ t = t_m \end{cases}$$

These are just some of the interesting properties. The others, equivalent to ones of the Wigner distribution, can be easily generalized[3].

IV. NUMERICAL RESULTS

Consider a sinusoidal frequency modulated signal (whose instantaneous frequency is highly nonlinear):

$$f(t) = e^{ja\cos(bt)} \qquad a = 8, \quad b = \pi.$$

We calculated the PWD using rectangular and Hanning windows, as well as the LWD (for $L = 2$ and $L = 4$) with the same windows and the same number of samples. The improvement by the proposed distribution is clearly shown in Figs 1., 2. and 3.

V. CONCLUSION

The comparison of commonly used time-frequency distribution with the distribution having the ideal instantaneous frequency representation is done. It is shown that the WD is the best among them. A generalization of the WD (LWD) is introduced. The LWD offers better results than the WD while preserving good properties of the WD. The computation time is slightly increased. The results are demonstrated on the numerical example with the frequency modulated signals.

[3]In the last three years (after this paper was submitted and undergone the first revision), the Higher order time-varying spectra have become a hot topic in the time frequency analysis, [11,13,14,15,17,19,21,22]. It turns out that the various reduced forms of the higher order time-varying spectra, [13,15,17], are just the special cases (with $L = 2$) of the distribution proposed here as the L-Wigner distribution. We have investigated the Higher order spectra, as well, and showed that generally, the L-Wigner distribution (in its dual form) is optimal in the analysis of multicomponent signals using the Wigner higher order spectra [19,22]. We have also defined the Multitime Wigner higher order distribution and derived the L-Wigner distribution (in form (12) presented and analyzed in this paper) as the optimal one in the case of multicomponent signals, [21]. In the meantime, a method for time-frequency analysis is proposed in [18]. This method, combined with the L-Wigner distribution provided a powerful tool for time-frequency analysis [21,22,23,24]. The L-Wigner distribution has been defined in the case of multidimensional signals as well [12]. Similar forms may be applied to the time-scale theory [23].

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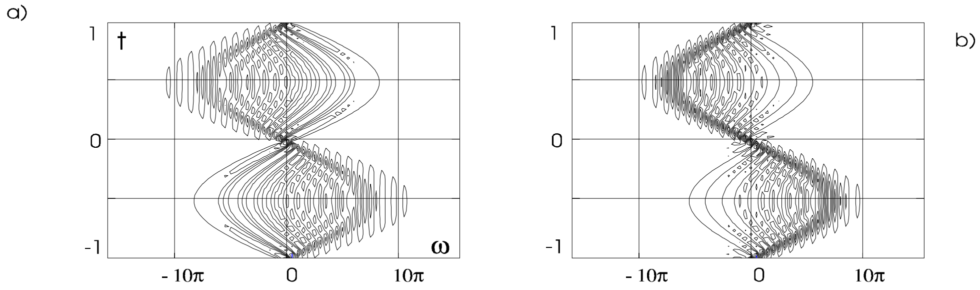

Fig. 1. Wigner distribution of a sinusoidal frequency modulated signal with: a) Rectangular window, b) Hanning window. $-1 < t < 1$, $-1 < \tau/2 < 1$, $N = 64$.

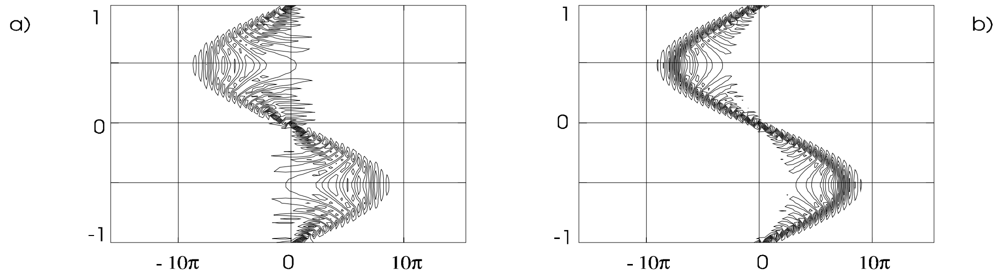

Fig. 2. L-Wigner distribution ($L = 2$) of a sinusoidal frequency modulated signal with: a) Rectangular window, b) Hanning window.

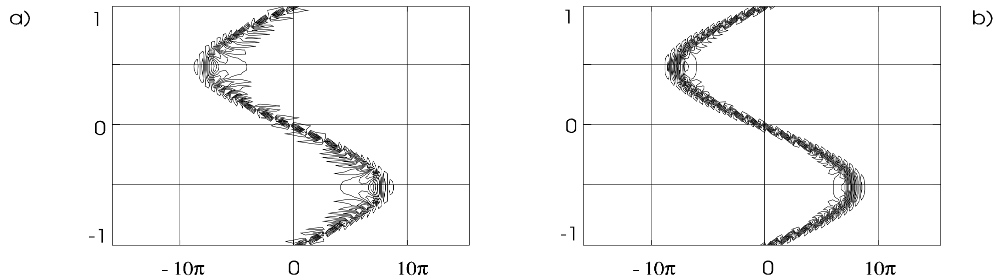

Fig. 3. L-Wigner distribution ($L = 4$) of a sinusoidal frequency modulated signal with: a) Rectangular window, b) Hanning window.

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