Time-Frequency Signal Analysis Based on the Windowed Fractional Fourier Transform

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Abstract— **A new signal-adaptive joint time-frequency distribution for the analysis of nonstationary signals is proposed. It is based on a fractional-Fourier-domain realization of the weighted Wigner distribution producing auto-terms close to the ones in the Wigner distribution itself, but with reduced cross-terms. Improvement over the standard time-frequency representations is achieved when the principal axes of a signal (defined as mutually orthogonal directions in the time-frequency plane for which the width of the signal's fractional power spectrum is minimum or maximum) do not correspond to time and frequency. The computational cost of this fractional-domain realization is the same as the computational cost of the realizations in the time or the frequency domain, since the windowed Fourier transform of the fractional Fourier transform of a signal corresponds to the short-time Fourier transform of the signal itself, with the window being the fractional Fourier transform of the initial one. The appropriate fractional domain is found from the knowledge of three second-order fractional Fourier transform moments. Numerical simulations confirm a qualitative advantage in the time-frequency representation, when the calculation is done in the optimal fractional domain. The approach can be generalized to the time-frequency distributions from the Cohen class.**

I. Introduction

Different types of joint time-frequency distributions are nowadays used in signal processing in order to extract the characteristic behavior of a signal. The advantages and disadvantages of most of the joint representations are well known. Thus, for example the appropriate Short-Time Fourier Transform (STFT) of a multi-component signal is almost free from cross-terms when the components do not overlap; meanwhile it does not stress well the auto-terms. On the other hand, the Wigner distribution (WD) [1] of such a signal suffers from the cross-terms, which may even hide some of the auto-terms. The aim of reduced interference distributions from the Cohen class [2], [3], [4] is to find optimal representations that would significantly decrease the cross-terms without degrading the auto-terms. Since frequently-used distributions from the Cohen class, such as for example the Choi-Williams, Bertrand, Butterworth, and Born-Jordan distributions, were designed for a general signal, they do not correspond to the optimal signal representation. In order to construct an optimal distribution, we have to adapt the distribution's kernel to a given signal [5], [6], [7], [8], [9]. Moreover, the adaptation should not be computationally too consuming, with minimum possible knowledge about the signal to be analyzed.

A method for time-frequency analysis of nonstationary signals, referred to as the S-method (SM), has been proposed in [10]. Its application produces the weighted Wigner distribution, or smoothed interferogram. The resulting distribution is of the WD form, with significantly reduced cross-terms of multi-component signals, while the auto-terms are close to those in the pseudo WD. This method is based on the STFT, in the initial step. The same method for signal analysis has been applied in several variants, in constructing appropriate time-frequency representations [11], [12], [13]. One of these forms combines the STFT values along frequency, for a given time instant, while the other is based on calculation in the time direction, for a given frequency. Both the efficiency of convergence towards the WD auto-terms and the cross-terms reduction depend on the orientation of the auto-terms

Signal Processing, vol.83, No.11, Nov. 2003.

in the time-frequency plane. In a general case the auto-terms might be oriented in a direction on some angle in the time-frequency plane, in which case the axes of minimum (maximum) signal width do not correspond to time or frequency. These rotated axes, corresponding to the minimum (maximum) signal width, will be referred to as the principal axes. Rotation of the time-frequency distribution kernels has been proposed in [14] in order to align the kernels' preferred axes to the signal's principal axes. The resulting time-frequency representations show a better reduction of cross-terms without too severely degrading the auto-terms than the corresponding, original time-frequency representations.

In this paper we introduce a form of the SM application on signal analysis in the fractional (mixed time-frequency) domain. In order to derive it, the STFT of a fractionally Fourier transformed signal has to be calculated. Since the STFT of a signal's fractional FT corresponds to the STFT of the signal itself with the window being the fractional FT of the initial one, with a subsequent rotation of the coordinate system, the STFT in the most appropriate fractional domain can be performed without significant additional computational costs. As an example, a modified Gaussian window for the STFT in the fractional FT domain is derived. After we get the STFT in the optimal fractional domain, the standard very simple implementation of the mentioned SM is performed. As a result of this method application we obtain a distribution which preserves the WD auto-terms and almost cancels the cross-terms.

In order to find the fractional domain associated with the principal axes, and to find the corresponding STFT, the analysis of fractional FT moments is applied. In particular, we suppose that an optimal fractional domain corresponds to minimum signal width, i.e., minimum second-order fractional FT moment. Calculation of this moment can be done analytically, based on three known moments for different fractional FT domains. The proposed approach is demonstrated on examples.

II. Short-time Fourier transform in the fractional FT domain

The STFT has been introduced for better time-localization of the frequency contents of a signal $x(t)$, by using a suitable window $g(t)$:

$$ST_x(t,f)=\int_{-\infty}^{\infty} x(t+t_o)g^*(t_o)e^{-j2\pi t_o f}dt_o. \quad (1)$$

Certainly, for filtering a pure sinusoidal signal, one needs a wide window, while for filtering of a delta-pulse like signal, a narrow window has to be applied. This rule also holds for the analysis of very wide-spread and very narrow signals, respectively. So, we can adjust the window if the signal shape is known. Suppose now that the minimum signal width does not correspond to the time or the frequency direction, as is the case for the signal illustrated in Fig. 1. Then we can see that an affine transformation of the phase plane leads to an optimal (for example, minimum width) signal representation. In this paper we restrict ourselves to pure coordinate rotation. In general the performance of filtering operations in the fractional Fourier domain was proposed in [15].

In order to represent a signal in a new coordinate system, we use the fact that rotation in the time-frequency plane corresponds to fractional FT of the signal. The fractional FT of a function $x(t)$ can be defined as [16]

$$R_x^{\alpha}(u) = X_{\alpha}(u) = \int_{-\infty}^{\infty} K(\alpha,t,u)x(t)dt, \quad (2)$$

where the kernel $K(\alpha,t,u)$ is given by

$$K(\alpha,t,u) = \frac{e^{j\alpha/2}}{\sqrt{j\sin\alpha}}e^{j\pi\frac{(t^2+u^2)\cos\alpha-2tu}{\sin\alpha}}. \quad (3)$$

Note that, in particular, $X_0(u) = x(u)$, $X_{\pi}(u) = x(-u)$, and that $X_{\pi/2}(u)$ corresponds to the normal FT of $x(t)$. Note moreover that, with the rotation-type relationship (cf. Fig. 1)

$$\begin{pmatrix} t \\ f \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (4)$$

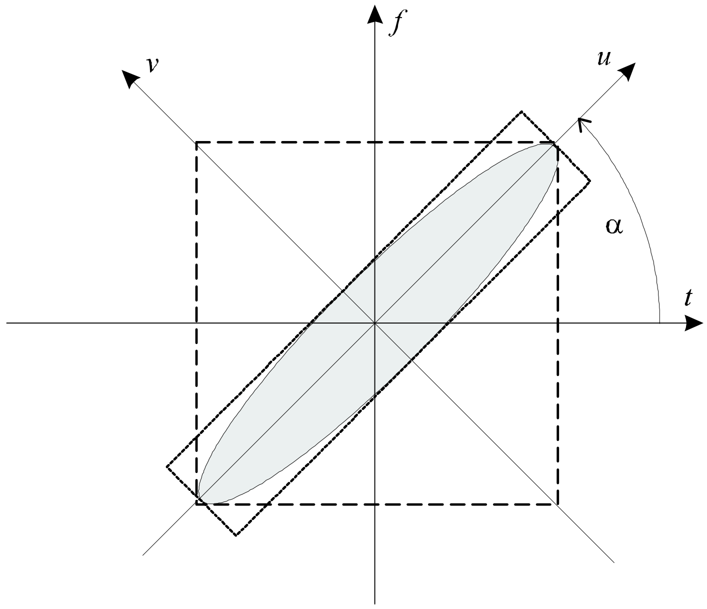

Fig. 1. Illustration of a signal whose principal axis in the time-frequency plane does not correspond to either time or frequency.

we have the following relationship for the fractional FT kernel:

$$K(\alpha, t_o, u-u_o)e^{j2\pi u_o v}e^{-j\pi uv}$$

$$= \left[K(-\alpha, u_o, t-t_o)e^{j2\pi t_o f}e^{-j\pi tf}\right]^*. \quad (5)$$

Let us consider the STFT $ST_x^\alpha(u,v)$ in the fractional domain α of a signal $x(t)$, defined as the STFT $ST_{X_\alpha}(u,v)$ of the fractional FT $X_\alpha(u)$ with the window $g(u)$:

$$ST_x^\alpha(u,v) = ST_{X_\alpha}(u,v)$$

$$= \int_{-\infty}^{\infty} X_\alpha(u+u_o)g^*(u_o)e^{-j2\pi u_o v}du_o \quad (6)$$

$$= \int_{-\infty}^{\infty} R_x^\alpha(u+u_o)\left[R_g^0(u_o)\right]^* e^{-j2\pi u_o v}du_o.$$

From relationship (5) we get (cf. [16, Section IV])

$$e^{-j\pi uv}\int_{-\infty}^{\infty} R_x^\alpha(u+u_o)\left[R_g^0(u_o)\right]^* e^{-j2\pi u_o v}du_o$$

$$=e^{-j\pi tf}\int_{-\infty}^{\infty} R_x^0(t+t_o)\left[R_g^{-\alpha}(t_o)\right]^* e^{-j2\pi t_o f}dt_o, \quad (7)$$

and from the latter identity we conclude that the STFT $ST_x^\alpha(u,v)$ in the fractional domain α can as well be calculated directly as a normal STFT of the signal $x(t)$ while using a window that is the fractional FT of the initial window $g(t)$, followed by the rotation (4) of the coordinate system:

$$ST_x^\alpha(u,v) =$$

$$e^{j\pi(uv-tf)}\int_{-\infty}^{\infty} x(t+t_o)\left[R_g^{-\alpha}(t_o)\right]^* e^{-j2\pi t_o f}dt_o \quad (8)$$

where u, v and t, f are related by (4).

Let us consider the Gaussian window $g(t) = \exp(-\pi c t^2)$, for which the fractional FT reads

$$R^\alpha_{\exp(-\pi ct^2)}(u) =$$

$$\frac{e^{j\alpha/2}}{\sqrt{\cos\alpha + jc\sin\alpha}} e^{-\pi c u^2 \frac{1+\tan^2\alpha - j(c-c^{-1})\tan\alpha}{1+c^2\tan^2\alpha}} \quad (9)$$

In the particular case $c = 1$, the Gaussian window is an eigenfunction of the fractional FT and the filtering with such a window in the fractional domain corresponds to a rotation of the STFT representation. In general there is a large class of window functions – eigenfunctions – of the fractional FT, which produce a rotation of the STFT for all or only a certain number of angles α. Thus, for example, the Hermite-Gauss functions $\Psi_n(t) = \exp(-\pi t^2)H_n(\sqrt{2\pi}t)$, where $H_n(t)$ are the Hermite polynomials, are eigenfunctions of the fractional FT for any angle α, while the function $\sum_{n=0}^{\infty} a_{L+Mn}\Psi_{L+Mn}(t)$, where a_n are arbitrary coefficients, is an eigenfunction of the fractional FT only for $\alpha = 2\pi k/M$.

Consider first a very simple case of the linear-FM signal

$$x(t) = e^{\left(j\pi pt^2 + j2\pi qt\right)} \qquad (10)$$

and assume that the Gaussian window $\exp(-\pi ct^2)$ would be the optimal window for filtering the pure harmonic signal $\exp(j2\pi qt)$. In order to find the optimal parameters of the Gaussian window (9) for filtering the linear-FM signal (10), we will turn to the fractional Fourier domain. If we apply the modulation theorem [16]

$$R^{\alpha}_{x(t)\exp(j2\pi qt)}(u) = R^{\alpha}_{x(t)}(u - q\sin\alpha)e^{j2\pi q\cos\alpha(u - q\sin\alpha/2)}$$

to the fractional FT of the chirp signal $\exp(j\pi pt^2)$,

$$\frac{e^{j\alpha/2}}{\sqrt{\cos\alpha}\sqrt{1 + p\tan\alpha}}\, e^{j\pi u^2 \frac{p-\tan\alpha}{1+p\tan\alpha}},$$

we conclude that the fractional FT of the linear-FM signal (10) becomes a pure harmonic signal with frequency $q\cos\alpha$ for the fractional angle $\alpha = \arctan p$, and a delta-pulse located at the position $q\sin\alpha$ for the fractional angle $\alpha = \pi/2 + \arctan p$. By returning to the time domain we obtain that the optimal Gaussian window for the linear-FM signal (10) takes the form

$$\begin{aligned}\left[R_g^{-\alpha}(u)\right]^* &= R^{\alpha}_{g^*}(u) \\ &= Ae^{-\pi cu^2 \frac{1+p^2 - j(c-c^{-1})p}{1+c^2p^2}}, \end{aligned} \qquad (11)$$

cf. Eq. (9) for the fractional angle $\alpha = \arctan p$.

In real-world problems a signal is not pure linear-modulated. It can even be multicomponent. Nevertheless, if the instantaneous frequencies of the signal components are changing slowly in the direction of a certain line in the time-frequency plane (we will refer to this line as the principal axis), we can find fractional domains where the signal is better concentrated or more spread. In order to find these fractional domains with minimum computational costs, leading to possible improvements in time-frequency representations, we will use the fractional FT moments of the signal.

III. Signal width estimation from fractional FT moments

It is known that the signal width in the time or the frequency domain can be estimated from its second-order central moments. Analogously, the signal width in the fractional domain is related to the second-order central fractional FT moments [17].

The second-order central fractional FT moment p_α is defined by

$$p_\alpha = \int_{-\infty}^{\infty} |R^{\alpha}_x(t)|^2 (t - m_\alpha)^2 dt = (w_\alpha - m_\alpha^2), \qquad (12)$$

where the first-order moment

$$m_\alpha = \int_{-\infty}^{\infty} |R^{\alpha}_x(t)|^2 t dt$$

is related to the center of gravity of the fractional power spectrum and where

$$w_\alpha = \int_{-\infty}^{\infty} |R^{\alpha}_x(t)|^2 t^2 dt$$

is the second-order moment. The first-order moment m_α in a fractional domain defined by an arbitrary angle α can be calculated from the relationship

$$m_\alpha = m_0\cos\alpha + m_{\pi/2}\sin\alpha, \qquad (13)$$

where m_0 and $m_{\pi/2}$ are the first-order moments in the time and the frequency domain, respectively. Also, any second-order moment w_α can be obtained from three other moments w_β, w_γ, and w_μ, say, if the angles β, γ, and μ are not the same, and the difference between them is not equal to π [17]. Let us choose three second-order moments: w_0, $w_{\pi/2}$, and $w_{\pi/4}$. Then using the results from [17], we have:

$$w_\alpha = w_0 \cos^2 \alpha + w_{\pi/2} \sin^2 \alpha$$
$$+[w_{\pi/4} - (w_0 + w_{\pi/2})/2] \sin 2\alpha. \quad (14)$$

Taking into account Eqs. (12), (13), and (14), we conclude that three fractional FT power spectra define all second-order central moments p_α, which characterize the signal widths in the corresponding fractional domains:

$$p_\alpha = (w_0 - m_0^2) \cos^2 \alpha + (w_{\pi/2} - m_{\pi/2}^2) \sin^2 \alpha$$
$$+[w_{\pi/4} - m_0 m_{\pi/2} - (w_0 + w_{\pi/2})/2] \sin 2\alpha$$
$$= p_0 \cos^2 \alpha + p_{\pi/2} \sin^2 \alpha$$
$$+[w_{\pi/4} - m_0 m_{\pi/2} - (w_0 + w_{\pi/2})/2] \sin 2\alpha. \quad (15)$$

In order to find the fractional domain where the signal has an extremal (minimum or maximum) width, we study the behavior of the derivatives of p_α. It is easy to see from Eq. (15) that the first derivative of p_α,

$$\frac{dp_\alpha}{d\alpha} = (p_{\pi/2} - p_0) \sin 2\alpha$$
$$+ [2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})] \cos 2\alpha,$$

equals zero for those angles α_e for which

$$\tan 2\alpha_e = \frac{2(w_{\pi/4} - m_0 m_{\pi/2}) - (w_0 + w_{\pi/2})}{p_0 - p_{\pi/2}}. \quad (16)$$

Since the fractional FT is periodic in α with period 2π (except for a possible factor -1) and satisfies the half-period relation $R_x^{\alpha+\pi}(t) = R_x^\alpha(-t)$, the signal width takes a minimum and a maximum value once over the region $\alpha \in [0, \pi)$. From the behavior of the second derivative of p_α for $\alpha = \alpha_e$, $d^2 p_\alpha / d\alpha^2 \,|_{\alpha=\alpha_e} = 2(p_{\pi/2} - p_0)/\cos 2\alpha_e$, we conclude that the signal reaches its minimum width for that value α_e for which $\cos 2\alpha_e$ has the same sign as $p_{\pi/2} - p_0$; the other value of α_e in the interval $[0, \pi)$ then corresponds to the maximum width. Thus, the appropriate fractional domain where the signal is best concentrated or most widely spread, can be found from the knowledge of only three fractional power spectra.

IV. S-method (SM) in the fractional domain

In the previous sections we have discussed the method how to perform the STFT in the most optimal way for a given signal. It can be achieved by choosing an appropriate window and an appropriate fractional domain. In this section we consider the discrete realization procedure, according to the SM, which leads to a representation close to the sum of the WDs of each signal component separately.

Consider a multi-component signal

$$x(t) = \sum_{i=1}^{M} x_i(t).$$

Its pseudo WD, defined by

$$PWD_x(t, f) = \int_{-\infty}^{\infty} x(t + \tau/2) x^*(t - \tau/2)$$
$$\times\, g^*(\tau/2) g\,(-\tau/2)\, e^{-j2\pi\tau f} d\tau,$$

has the form

$$PWD_x(t, f) =$$
$$\sum_{i=1}^{M} PWD_{x_i}(t, f) + \sum_{i=1}^{M} \sum_{k=1, k\neq i}^{M} PWD_{x_i, x_k}(t, f);$$

note that the normal WD arises for $g(t) = 1$. In most applications, the aim of time-frequency analysis is to get a distribution that contains only the sum of auto-terms $\sum_{i=1}^{M} PWD_{x_i}(t, f)$ without the cross-terms $PWD_{x_i, x_k}(t, f)$. It is also known that the WD, among all other quadratic signal-dependent, time-frequency distributions, has the best auto-term concentration. In most cases the reduced interference distributions are obtained at the cost of significant auto-terms degradation.

The pseudo WD can also be expressed in terms of the STFT as

$$PWD_x(t,f) = \int_{-\infty}^{\infty} ST_x(t,f+\theta/2)ST_x^*(t,f-\theta/2)d\theta.$$

Based on this definition of the pseudo WD, the SM for time-frequency analysis is based on the relation [12], [18]

$$P_x(t,f) = \int_{-\infty}^{\infty} ST_x(t,f+\theta/2)z(\theta)ST_x^*(t,f-\theta/2)d\theta, \quad (17)$$

where the additional frequency window $z(\theta)$ is used to exclude the interference pattern between frequency-misaligned versions, while it should be wide enough to provide complete integration over auto-terms of the STFT $ST(t,f)$. Note that, in comparison to [10], [11], [12], we use a slightly different definition for the SM-based analysis, in order to get a nicer equivalence to the WD. It is easy to see that if $z(\theta) = 1$ we get the pseudo WD, while for $z(\theta) = \delta(\theta)$ we obtain the time-varying spectrogram. If the width of $z(\theta)$ is somewhere in between, we can expect, as it was proved in [11] and [12], that the corresponding distribution combines the nice properties of both the spectrogram and the WD. It is known that the spectrogram does not suffer from cross-terms, in contrast to the WD where the cross-terms are very emphatic. On the other hand, the spectrogram has a significant leakage due to the window usage, which is less exhibited in the case of the WD. By choosing an appropriate function $z(\theta)$, the sharpness of the WD can be preserved and the cross-terms will be reduced or even completely removed. For that to be the case, the lag window in the STFT has to be such that the components of the STFT are not far from the instantaneous frequencies of the signal components, in order to obtain fast convergence inside $z(\theta)$.

An SM application could also be based on time-direction combined STFTs. It is then based on the form [11], [12]

$$\int_{-\infty}^{\infty} ST_x(t+\theta/2,f)z(\theta)ST_x^*(t-\theta/2,f)e^{-j2\pi f\theta}d\theta \quad (18)$$

Which one of the previous two forms (17) and (18) would produce better results depends on the signal. If the auto-terms in the STFT are well concentrated along the frequency direction, then the form (17) would be the better choice, and vice versa.

As it has been discussed in the previous section, concentration of the STFT can be improved for signals whose principal axes are not the time and frequency axes, but fractional axes in directions defined by α and $\alpha+\pi/2$. We have already found that for a given signal there exists a fractional domain where the STFT can be performed in an optimal way. Finding the domain where the signal is best concentrated is based on the fractional FT moments. We can expect that the application of the SM in that particular domain will be the most efficient one. There, the FT of the signal's fractional FT occupies the narrowest range. The SM in this fractional domain is based on [19]

$$P_x^\alpha(t,f) = \int_{-\infty}^{\infty} ST_x^\alpha(u,v+\theta/2)z(\theta)ST_x^{\alpha *}(u,v-\theta/2)d\theta, \quad (19)$$

where $ST_x^\alpha(u,v)$ and the relation between (u,v) and (t,f) are defined by Eqs. (6) and (4), respectively. Using the rotational properties of the STFT, Eq. (7), we can rewrite Eq. (19) as

$$P_x^\alpha(t,f) = \int_{-\infty}^{\infty} ST_x(t+(\theta\sin\alpha)/2, f+(\theta\cos\alpha)/2)z(\theta) \times e^{-j2\pi f\theta\sin\alpha}ST_x^*(t-\frac{\theta\sin\alpha}{2}, f-\frac{\theta\cos\alpha}{2})d\theta, \quad (20)$$

from which it is clear that the SM in the fractional domain corresponds to the

SM applied simultaneously in the time and frequency domains. The two special cases (17) and (18) follow as special cases from Eq. (20) for $\alpha = 0$ and $\alpha = \pi/2$, respectively.

V. Discrete form

The analog form (20) suggests that the discrete form of the SM application in an arbitrary domain can be calculated based on the original signal's STFT. However, the values of the STFT arguments do not correspond to the discretization grid: the STFT values should be calculated by using interpolation for each time-frequency point, and a given α. A much simpler calculation is based on Eqs. (6) or (8) and (19). After the angle α has been determined, for which the second-order fractional FT moment is minimum, see Eq. (16), the discrete fractional FT of the signal $X_\alpha(n)$ (or of the window) is calculated. The discrete STFT then reads [cf. Eq. (6)]

$$ST_x^\alpha(n,k) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} X_\alpha(n+m)g^*(m)e^{-\frac{jmk2\pi}{N}}.$$

The discrete SM is of the form [cf. Eq. (19)]

$$P_x^\alpha(n,k) = \sum_{m=-N/2}^{N/2-1} ST_x^\alpha(n,k+m)z(m)ST_x^{\alpha*}(n,k-m)$$

or

$$P_x^\alpha(n,k) = |ST_x^\alpha(n,k)|^2 + 2\,\mathrm{Re}\left\{\sum_{m=1}^{N_z} ST_x^\alpha(n,k+m)ST_x^{\alpha*}(n,k-m)\right\}, \tag{21}$$

where we have tacitly assumed a real, rectangular window $z(m)$ with width $2N_z + 1$. Therefore, the SM-based calculation can be understood as calculation of the spectrogram in the domain defined by α, and its improving by terms $2\,\mathrm{Re}\{ST_x^\alpha(n,k+m)ST_x^{\alpha*}(n,k-m)\}$ towards the rotated WD quality of auto-terms. Taking just a few of these spectrogram-correcting terms around the time-frequency point under consideration, we immediately start improving the auto-term concentration, while the cross-terms will appear when we start taking values from other auto-terms. Taking $N_z = N/2$ we get the rotated WD.

VI. Numerical example

Consider the signal

$$x(t) = e^{-(3t)^8}\{e^{j(192\pi t^2 - 8\cos(4\pi t)/\pi)} + e^{j(64\pi t^2 + 8\cos(4\pi t)/\pi)}\}$$

sampled at $T = 1/256$. A Hanning lag window, with $N_w = 128$ samples, is used for the STFT calculation. The values of the second-order central moments [normalized with respect to the zero-order moment $\int_{-\infty}^{\infty} |x(t)|^2 dt$] are $p_0 = 1$, $p_{\pi/2} = 1.38$, and $p_{\pi/4} = 0.07$. According to Eq. (16), and using the fact that $p_0 < p_{\pi/2}$, we get $\alpha_e = 41°$. The second-order moment in this direction is smaller than in any other direction: $p_{41°} = 0.057$, while the second-order moment in the orthogonal direction is the largest: $p_{-49°} = 2.01$. Now the fractional FT of the signal for the angle $\alpha = \alpha_e - \pi/2 = -49°$ can be calculated by using the discrete fractional FT algorithms, or just by using the inversion property of the rotated WD. The next step is to calculate the STFT of the fractional FT and to use it in Eq. (21).

The results of this analysis are presented in Fig. 2. The standard WD is shown in Fig. 2a. The SM-based distribution, calculated by the standard definition, i.e., along the frequency axis, with $N_z = 10$ correcting terms, is presented in Fig. 2b. We see that some cross-terms already appear, although the auto-terms are still very different from those in the WD in Fig. 2a. The reason lies in the very significant spread of one component along the frequency axis. Fig. 2c shows the WD of the fractional FT for $\alpha = -49°$, obtained as the optimal angle for this signal; note that it is just a rotated version of the original WD. The SM-based distribution on the fractional FT is presented in Fig. 2d. We can see that, as a consequence of the high concentration of the components along the optimal fractional angle, we almost achieved the goal of getting the auto-terms of the WD without any cross-terms.

Note that if the signal is already well-concentrated in time or in frequency, then the proposed procedure will also produce the standard calculation directions as special cases.

Similar results are obtained with the signal

$$x(t) = e^{-(3t)^8}[e^{(j\phi(t)+50\pi t)} + e^{(j\phi(t)-50\pi t)}]$$

$$\phi(t) = \int_{-\infty}^{t} 15\,\pi \operatorname{arcsinh}(100t)\,dt$$

and the same discretization parameters as in the previous example, see Fig. 3.

VII. Generalization

The presented approach can be generalized to the kernels from the Cohen class of time-frequency representations [2], [3], [4],

$$C_x(t,f) = \iint_{-\infty}^{\infty} \Phi(t_o,f_o)W_x(t-t_o,f-f_o)dt_o df_o, \quad (22)$$

where $W_x(t,f)$ is the WD and $\Phi(t,f)$ is the kernel in the time-frequency domain.

In many cases the kernel $\Phi(t,f)$ shows a preferred behavior in the time and/or the frequency direction. The degree of cross-term reduction (and degrading of the auto-terms) then depends on the way in which the WD is oriented in the time-frequency plane. If the orientation is along the time and/or the frequency direction, the kernel may act as expected; in the case of a different orientation, the effect of the kernel is not optimal [7]. In [14] we therefore suggested to rotate the kernel in such a way that its preferred axes coincide with the principal axes of the WD. Note that, although the rotated distributions may not satisfy the common marginal properties, they satisfy generalized ones [9].

In the case of the SM application in the fractional domain, with $C_x(t,f) = P_x^{\alpha}(t,f)$, the kernel in the Wigner domain reads

$$\Phi(t,f) = W_g(-t,-f)\,Z(-[t\cos\alpha - f\sin\alpha]),$$

where $W_g(t,f)$ is the WD of the window $g(t)$ and $Z(f)$ is the FT of the window $z(\theta)$, while its double Fourier transform

$$\bar{\Phi}(\tau,\nu) = \iint_{-\infty}^{\infty} \Phi(t,f)e^{-j2\pi(\nu t - f\tau)}dtdf,$$

which acts as a multiplier function in the ambiguity domain, takes the form

$$\bar{\Phi}(\tau,\nu) = \int_{-\infty}^{\infty} A_g(-\tau+\theta\sin\alpha, -\nu+\theta\cos\alpha)z(\theta)d\theta,$$

where $A_g(\tau,\nu)$ is the ambiguity function of the window $g(t)$. Note that for $z(\theta) = \delta(\theta)$ the multiplier function reduces indeed to the one of the spectrogram, $\bar{\Phi}(\tau,\nu) = A_g(-\tau,-\nu)$, as we remarked before. For $z(\theta) = 1$ it reduces to

$$\bar{\Phi}(\tau,\nu) = \int_{-\infty}^{\infty} A_g(-\tau+\theta\sin\alpha, -\nu+\theta\cos\alpha)d\theta$$
$$= G_{-\alpha}(-[\tau\cos\alpha - \nu\sin\alpha]/2)$$
$$\times G_{-\alpha}^{*}([\tau\cos\alpha - \nu\sin\alpha]/2),$$

with $G_\alpha(u)$ being the fractional FT of the window $g(t)$; and for $\alpha = 0$ we then get again the pseudo WD with $G_0(t) = g(t)$. The details of the derivation are presented in the Appendix.

Other Cohen class distributions (Butterworth, generalized exponential, and Zhao-Atlas-Marks) have been treated in a previous paper [14]. For all these distributions, but in particular for the SM-based distribution, we conclude that the reduction of cross-terms without too severely degrading the auto-terms, is better for the aligned kernels than for the non-aligned ones.

VIII. Conclusion

A method for the analysis of nonstationary signals is presented. It is based on the representation of the signal in the fractional domain where it has minimum/maximum second-order fractional FT moments. The signal is represented in the new fractional time-frequency domain by the windowed fractional Fourier transform. Concentration of this transform is then improved by using the S-method. The theory is illustrated on two examples with signals which do and do not intersect in the time-frequency domain.

Appendix

The rotated kernel for SM-based distributions

From the WD definition we can write the relationship

$$x(t_1)x^*(t_2)$$

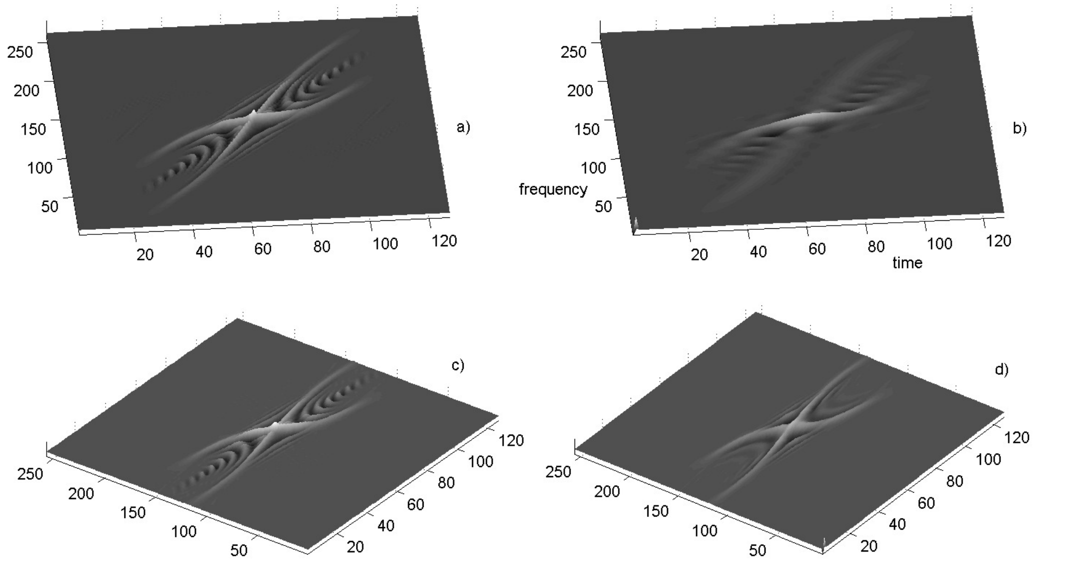

Fig. 2. a) Wigner distribution of the signal, b) The SM-based distribution calculated in the frequency domain, c) Rotated WD (WD of the fractional FT), d) The SM-based distribution calculated in the "optimal" fractional frequency domain.

$$= \int_{-\infty}^{\infty} W_x((t_1+t_2)/2, f) e^{j2\pi f(t_1-t_2)} df. \quad (23)$$

The STFT is [see Eq. (1)]

$$ST_x(t,f) = \int_{-\infty}^{\infty} x(t+\tau) g^*(\tau) e^{-j2\pi f\tau} d\tau. \quad (24)$$

The generalized SM-based distribution is defined by Eq. (20). After substitution of Eq. (24) into (20) we get

$$P_x^{\alpha}(t,f) =$$

$$\iiint_{-\infty}^{\infty} x(t+\theta\sin\alpha/2+t_1) x^*(t-\theta\sin\alpha/2+t_2)$$
$$\times\; g^*(t_1) g(t_2) z(\theta) e^{-j2\pi(f+\theta\cos\alpha/2)t_1}$$
$$\times\; e^{j2\pi(f-\theta\cos\alpha/2)t_2} e^{-j2\pi f\theta\sin\alpha} dt_1 dt_2 d\theta.$$

By using Eq. (23) and the substitutions $t_1 = t_0 - t/2 + \tau/2$, $t_1 = t_0 - t/2 - \tau/2$, and after some transformations, we get

$$P_x^{\alpha}(t,f) = \iint_{-\infty}^{\infty} W_x(t_o, f_o) dt_o df_o$$

$$\times \int_{-\infty}^{\infty} g^*(t_o - t + \frac{\tau}{2}) g(t_o - t - \frac{\tau}{2}) e^{j2\pi(f_o-f)\tau} d\tau$$

$$\times \int_{-\infty}^{\infty} z(\theta) e^{j2\pi\theta\{(f_o-f)\sin\alpha-(t_o-t)\cos\alpha\}} d\theta$$

$$= \iint_{-\infty}^{\infty} W_x(t_o, f_o) W_g^*(t_o - t, f_o - f)$$

$$\times\; Z([t_o - t]\cos\alpha - [f_o - f]\sin\alpha) dt_o df_o.$$

The kernel in the time-frequency domain (the smoothing function for the WD) is thus given by

$$\Phi(t,f) = W_g(-t,-f)\, Z(-[t\cos\alpha - f\sin\alpha]),$$

whereas in the ambiguity domain the kernel reads

$$\bar{\Phi}(\tau,\nu) = \iint_{-\infty}^{\infty} \Phi(t,f) e^{-j2\pi(\nu t - f\tau)} dt df$$

$$= \int_{-\infty}^{\infty} A_g(-\tau + \theta\sin\alpha, -\nu + \theta\cos\alpha) z(\theta) d\theta.$$

Note that for $z(\theta) = \delta(\theta)$ the multiplier function reduces to the case of the

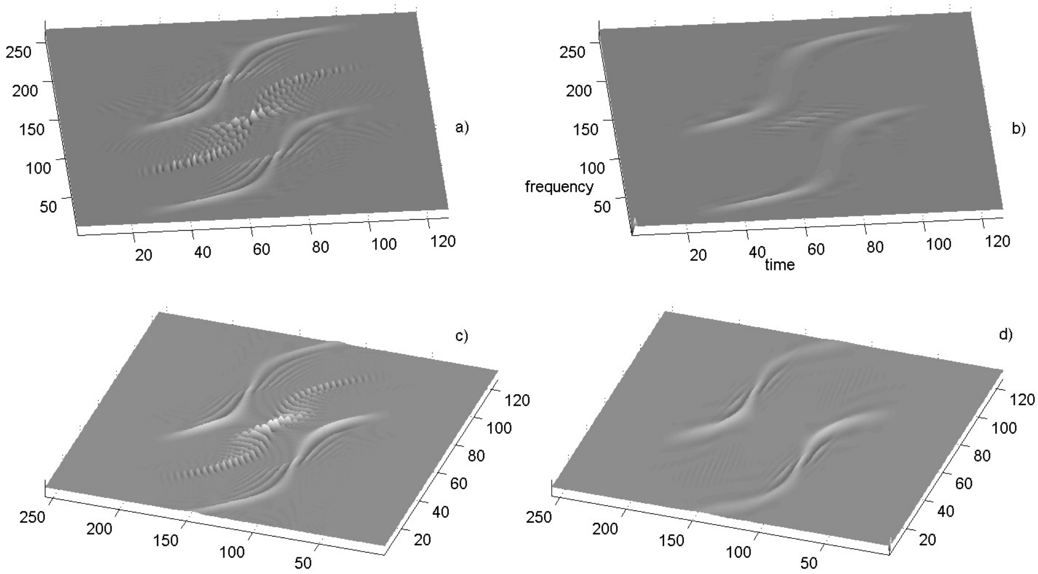

Fig. 3. a) Wigner distribution of the signal, b) The SM-based distribution calculated in the frequency domain, c) Rotated WD (WD of the fractional FT), d) The SM-based distribution calculated in the "optimal" fractional frequency domain.

spectrogram, $\bar{\Phi}(\tau,\nu) = A_g(-\tau,-\nu)$. On the other hand, for $z(\theta) = 1$ and $\alpha = 0$ we get the pseudo WD with $G_0(t) = g(t)$.

Acknowledgment

The work of LJ. Stanković is supported by the Volkswagen Stiftung, Federal Republic of Germany. The work of T. Alieva is supported by the Spanish Ministry of Science and Technology under the project TIC2002-01846.

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