Local Polynomial Wigner Distribution

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Abstract— **Local polynomial time-frequency transformes, introduced by Katkovnik, are extended, in this paper, to the Wigner distribution. These distributions may be efficiently realized, in the case of multicomponent signals, using the S-method.**

I. Local Polynomial Time-Frequency Transform

Local polynomial time-frequency transform (LPTFT) has been introduced by Katkovnik, [1], [2], as:

$$X(\vec{\omega},t)=\int_{-\infty}^{\infty} x(t+\tau)w(\tau)e^{-j\theta(\vec{\omega},\tau)}d\tau \quad (1)$$

where

$$\theta(\vec{\omega},\tau)= =\omega_1\tau+\omega_2\frac{\tau^2}{2!}+\omega_3\frac{\tau^3}{3!}+....+\omega_m\frac{\tau^m}{m!} \quad (2)$$

The LPTFT is an $m+1$ dimensional transform which concentrates at:

$$\vec{\omega}^*=(\phi'(t),\phi^{(2)}(t),...,\phi^{(m)}(t)) \quad (3)$$

where $\phi'(t)$ is the signal's instantaneous frequency, and $\phi^{(2)}(t),...,\phi^{(m)}(t)$ are its first, second and $(m-1)$ derivatives, respectively. If we are interested in the instantaneous frequency estimation, then $\omega_2,\omega_3,...,\omega_m$ should be considered as auxiliary variables and the estimation is based on the maxima of $X(\vec{\omega},t)$ over the space $\omega_2,\omega_3,...,\omega_m$. These maxima are now two-dimensional function of (ω_1,t), denoted by $M(\omega_1,t)$. Note that the conventional STFT is obtained at $\omega_1=\omega$, $\omega_2=\omega_3=...=\omega_m=0$. As opposite to the polynomial Wigner-Ville distributions [3], [4], [5], [6], the LPTFT is a linear transform with respect to the signal, what is very important property, but it is paid by a multidimensional form of the distribution, what make its realization complex. Discrete-time version of the LPTFT has been proposed and justified in [7]. The local polynomial periodogram (LPP) is given by:

Signal Processing, vol.59, No.1, May 1997.

$$I(\vec{\omega},t)=\left|X(\vec{\omega},t)\right|^2. \quad (4)$$

Integral of the LPTFT over ω_1 produces original signal, $x(t)=\int_{-\infty}^{\infty}X(\vec{\omega},t)d\omega_1/(2\pi w(0))$, while a form of Parseval's theorem may be written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(\vec{\omega},t)\right|^2 d\omega_1= \int_{-\infty}^{\infty}\left|x(t+\tau)w(\tau)\right|^2 d\tau \quad (5)$$

what demonstrates that the LPP may be treated as an energetic distribution. These are only some of the properties of the LPTFT and LPP. Further details may be found in [1], [2], [7].

II. Local Polynomial Wigner Distribution

Here, we introduce the local polynomial Wigner distribution (LPWD) as:

$$W(\vec{\omega},t)= \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})w_e(\tau)e^{-j\theta_o(\vec{\omega},\tau)}d\tau \quad (6)$$

where now we do not have (there is no need for) the even terms in $\theta(\vec{\omega},\tau)$. Instead of $\theta(\vec{\omega},\tau)$ we use $\theta_o(\vec{\omega},\tau)$ which is defined by:

$$\theta_o(\vec{\omega},\tau)= =\omega_1\tau+\omega_3\frac{\tau^3}{3!}+....+\omega_{2n+1}\frac{\tau^{(2n+1)}}{(2n+1)!} \quad (7)$$

Distribution (6) is concentrated at $\vec{\omega}^*=(\phi'(t),\phi^{\{3\}}(t),...,\phi^{(2n+1)}(t))$. Therefore, in this way we may achieve the instantaneous frequency estimation with a significantly reduced distribution dimension, i.e., reduced number of the auxiliary variables (instead of $m+1$ dimensional space we have $m/2+1$ dimensional one). For example, if we can expect that the signal's phase has significant derivatives up to the fourth order, then only ω_1 and ω_3 axes, i.e., $\theta_o(\vec{\omega},\tau) = \omega_1\tau + \omega_3\tau^3/3!$, may be used, instead of four coordinates, $\omega_1, \omega_2, \omega_3, \omega_4$, in $\theta(\vec{\omega},\tau) = \omega_1\tau + \omega_2\tau^2/2! + \omega_3\tau^3/3! + \omega_4\tau^4/4!$ in (1). If we consider only the coordinates with nonlinear τ (auxiliary variables for the frequency estimation), then, in this example, instead of three coordinates $(\omega_2, \omega_3, \omega_4)$, we need only one (ω_3).

Integral of the LPWD over ω_1 is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} W(\vec{\omega},t)d\omega_1 = |x(t)|^2 \tag{8}$$

for $w_e(0) = 1$. The LPWD is quadratic distribution with respect to signal, the same as the LPP is. But, the LPWD, in contrast to the LPP, exhibits cross-terms in the case of multicomponent signal, even if the signal components are far apart in the time-frequency plane. This drawback can be avoided using the S-method, [8], [9], [10], [11], [12], for the local polynomial Wigner distribution realization.

III. S-Method in the LPTFT

Local polynomial Wigner distribution may be realized without cross-terms, if the signal's component do not overlap in LPP, using the S-method in the form:

$$S(\vec{\omega},t) = \\ = \int_{-\infty}^{\infty} P(\theta)X(\vec{\omega}+\vec{\lambda},t)X^*(\vec{\omega}-\vec{\lambda},t)d\theta \tag{9}$$

where $\vec{\lambda} = (\theta, 0, 0, ...0)$. Window $P(\theta)$ controls the convolution in θ, which is one-dimensional variable along the first coordinate ω_1. In the case of a multicomponent signal

$$x(t) = \sum_{i=1}^{M} x_i(t) \tag{10}$$

all auto-terms are located at, and around, $\theta = 0$, while the cross-terms are dislocated from that point. Suppose that the LPTFT of the i-th component $X_{x_i}(\vec{\omega},t)$, $i = 1, 2, ..., M$, is concentrated along ω_1 around its instantaneous frequency $\phi_i'(t)$ within the region $\left|\omega_1 - \phi_i'(t)\right| < B$. All auto-terms in (9) are concentrated in the region $|\theta| < B$, while the cross-term between i-th and j-th components is located within $\left|\theta - [\phi_i'(t) - \phi_j'(t)]/2\right| < B$. If the width of low-pass frequency domain window $P(\theta)$, denoted by W_P, is large enough to provide integration over auto-terms, $W_P \geq B$, and if the auto-terms in the LPTFT are at least B apart we may get

$$S(\vec{\omega},t) = \sum_{i=1}^{M} W_{x_i}(\vec{\omega},t) \tag{11}$$

where $W_{x_i}(\vec{\omega},t)$ is the LPWD of the component $x_i(t)$. Using the variable width window $P(\theta)$ approach, described in [12], relation (11) holds with a less restrictive condition that the auto-terms do not overlap along ω_1 for any t, i.e., relation (11) holds if the LPP does not have cross-terms. Since the window $P(\theta)$ is usually quite narrow, this method is numerically very efficient, [9], [12]. Since all even terms $\theta(\vec{\omega},\tau)$ will be eliminated by convolution (9), then we can calculate the LPTFT, needed in (9), using only $\theta_o(\vec{\omega},\tau)$ defined by (7):

$$X_o(\vec{\omega},t) = \int_{-\infty}^{\infty} x(t+\tau)w(\tau)e^{-j\theta_o(\vec{\omega},\tau)}d\tau \tag{12}$$

i.e., by taking coordinates $\omega_2 = \omega_4 = \omega_{2n} = 0$ in (1). Inserting (12) into (9) we get the cross terms free distribution with as high concentration as in (4), applying only a half of the coordinates required in (1).

IV. Example

If we want to include terms in the phase expansion up to the fourth order term, then the S-method has the form:

$$S(\omega_1,\omega_3,t) = \\ \int_{-\infty}^{\infty} P(\theta)X_o(\omega_1+\theta,\omega_3,t)X_o^*(\omega_1-\theta,\omega_3,t)d\theta \tag{13}$$

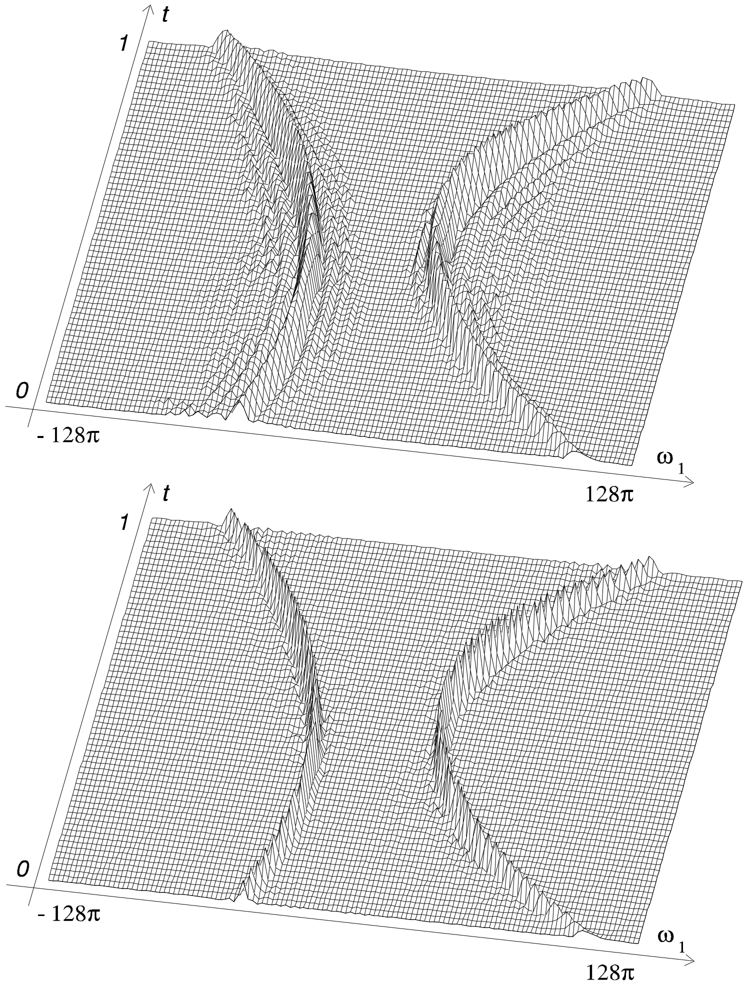

Fig. 1. Time-frequency representation of a multicomponent signal using the local polynomial approach: a) Maxima of the S-method, b) Maxima of the S-method along the lines where they are detected.

where

$$X_o(\omega_1, \omega_3, t) =$$

$$= \int_{-\infty}^{\infty} x(t+\tau)w(\tau)e^{-j(\omega_1\tau + \omega_3\tau^3/3!)}d\tau \quad (14)$$

Consider now a particular signal

$$x(t) = Ae^{j\phi(t)} =$$

$$= Ae^{j(at+bt^2/2+ct^3/3!+dt^4/4!)} \quad (15)$$

The LPTFT of $x(t)$, with $\theta_o(\vec{\omega}, \tau)$ as in (7), is

$$X_o(\omega_1, \omega_3, t) =$$

$$= x(t)W\left(\omega_1 - (a + bt + ct^2/2 + dt^3/3!)\right) *_{\omega_1}$$

$$FT\left\{e^{j[\phi^{(2)}(t)\tau^2/2+\phi^{(4)}(t)\tau^4/4!]}\right\}$$

$$\text{on } \omega_3 = \phi^{(3)}(t) = c + dt.$$

Note that, as compared to (1), we assumed $\omega_1 = \omega_4 = 0$, since these coordinates do not influence the result in the LPWD. The LPWD of signal (15) is given by:

$$W(\omega_1, \omega_3, t) =$$

$$= A^2 W_e\left(\omega_1 - (a + bt + ct^2/2 + dt^3/3!)\right) \quad (16)$$

$$\text{on } \omega_3 = \phi^{(3)}(t).$$

The same result will be obtained with $S(\omega_1, \omega_3, t)$ if the window $P(\theta)$ is wide enough to provide integration over the auto-terms along ω_1, for a given t. If we had used the LPTFT, given by (1), using all four coordinates we would get the same concentration as in (16):

$$|X(\omega_1, \omega_2, \omega_3, \omega_4, t)|^2 = = A^2 W^2 \left(\omega_1 - (a + bt + ct^2/2 + dt^3/3!)\right) \tag{17}$$

$$\text{along } \omega_2 = \phi^{(2)}(t),\ \omega_3 = \phi^{(3)}(t) \text{ and } \omega_4 = \phi^{(4)}(t).$$

As a numerical example, for a multicomponent signal, consider

$$x(t) = [e^{j(20\pi t + 100\pi(t-0.5)^3 - 15\pi(t-0.5)^4)} + +e^{-j(30\pi t + 50\pi(t-0.3)^3 - 20(t-0.3)^4)}]e^{-2(t-0.5)^2}$$

with the following parameters: time interval $0 \le t < 1$; sampling period $\Delta t = 1/128$; the Hanning window $w(\tau)$ of the unity width; and the rectangular window $P(\theta)$ with variable width and reference value at 4% of the maximal periodogram value for a given t. Coordinate ω_3 is taken from -1000π to 1000π with a step of 100π. Maximal values of the S-method (calculated using (13) via (14), [8], [9], [10], [11], [12]) over ω_3, for each value of t, $SM(\omega_1, t) = \max_{\omega_3}\{S(\omega_1, \omega_3, t)\}$, is presented in Fig.1a. Note that this is the best possible presentation with respect to the auto-term values, since it takes its maxima, but with respect to the side lobes this is the worst possible presentation, since the maximal values of the side lobes are also taken for presentation in Fig.1a, [1], [2]. Fig.1b. presents the values of $S(\omega_1, \omega_3, t)$ taken along the lines ω_3 where the maxima are detected (for positive ω_1 semiplane, where the maxima of signal's first component are detected, and for negative ω_1 semiplane, the lines ω_3 at which the maxima of the second component are detected). This second approach is more illustrative, but less practically applicable, since it requires a priori knowledge of the ranges within which the signal component instantaneous frequencies vary.

V. Conclusion

A distribution based on the local polynomial time-frequency transform (LPTFT) and Wigner distribution is introduced. This distribution (LPWD) may produce very high signal concentration at the instantaneous frequency with a reduced problem dimension. Using the S-method an efficient realization of this distribution is presented.

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