

Influence of High Noise on the Instantaneous Frequency Estimation Using Quadratic Time-Frequency Distributions

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Abstract— Analysis of time-frequency (TF) distributions, as the instantaneous frequency (IF) estimators, for small noise, has been recently done. In this letter, we extend the analysis to the high noise. This noise causes a specific error, which can dominate over all other studied errors. The crucial parameter is the ratio of auto-term (AT) magnitude and distribution standard deviation.

I. INTRODUCTION

The IF estimation by using TF analysis is based on the detection of a distribution maxima positions [1]-[4]. The sources of estimation error are: 1) bias; 2) random deviation of the maxima within the auto-term, caused by the small noise (this noise can make some of the AT points surpass the value of true maximum at the IF); 3) large random deviations due to false maxima detection outside the AT. It happens when the noise is so high that some of the distribution values outside the AT surpass the values inside the AT. This error can significantly degrade the estimation. The first two kinds of errors have been studied in detail in [2], [3]. The paper is focused on the analysis of the third source of error. The IF estimation based on the Wigner distribution (WD) is analyzed in Section II. The Cohen class (CD) of distributions is considered in Section III.

II. ESTIMATION ERROR

Consider the WD:

$$WD_{xx}(n, k) =$$

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$$\sum_{m=-N/2}^{N/2-1} x(n+m)x^*(n-m)e^{-j4\pi\frac{mk}{N}} \quad (1)$$

of a frequency modulated (FM) signal $x(n) = f(n) + \nu(n) = Ae^{j\phi(n)} + \nu(n)$, corrupted by a Gaussian white complex noise with variance $2\sigma^2$. For a given instant n the IF is estimated according to the WD maximum position $\hat{k} = \arg\{\max_k WD_{xx}(n, k)\}$. For the analysis of high noise influence, note that the WD mean value is $WD_{ff}(n, m) + 2\sigma^2$, while the variance is $\sigma_{WD}^2 = 4N\sigma^2(A^2 + \sigma^2)$ [5]. The constant factor $2\sigma^2$ in the mean value will be omitted. Since there is a large number of terms in sum (1), we will assume that the central limit theorem may be applied to the WD values. Thus, they are Gaussian in nature, with $\mathcal{N}(0, \sigma_{WD})$ outside the AT, and $\mathcal{N}(A_{WD}, \sigma_{WD})$ within the AT. Here, A_{WD} is the AT maximal value for given n , $A_{WD} = \max_k\{WD_{ff}(n, k)\}$. The above assumption has been statistically checked.

The probability density function (pdf) for the WD values at the AT is then:

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma_{WD}} e^{-(\xi - A_{WD})^2 / 2\sigma_{WD}^2}. \quad (2)$$

The WD outside the AT assumes a value greater than Ξ , with probability:

$$\begin{aligned} Q(\Xi) &= \frac{1}{\sqrt{2\pi}\sigma_{WD}} \int_{\Xi}^{\infty} e^{-\xi^2 / 2\sigma_{WD}^2} d\xi \\ &= 0.5\text{erfc}\left(\frac{\Xi}{\sqrt{2}\sigma_{WD}}\right). \end{aligned}$$

Probability that any WD value, at M points outside the AT, is greater than Ξ is:

$$G(\Xi) = 1 - (1 - Q(\Xi))^M. \quad (3)$$

When a WD value outside AT surpasses the value within AT then a large IF estimation error occurs. From (2) and (3), follows that probability for its occurrence is:

$$P_E = \int_{-\infty}^{\infty} G(\xi)p(\xi)d\xi. \quad (4)$$

This error is then of impulse nature, and its values are uniformly distributed over the entire frequency interval.

Relation (4) is illustrated on an example with LFM signal $f(n) = A \exp(jan^2/2)$. In order to avoid the discretization error the value of a is chosen so that the exact IF lies along the frequency grid. This a and large N in (1) produce highly concentrated AT and eliminate, for this signal, the errors within the AT. Then $A_{WD} = NA^2$, and $M = N - 1$. In this way the only remaining error is due to the false AT position detection, whose analysis is the primary goal of this paper. The histograms of the WD values along the IF, $p(\xi)$, and outside the AT, $q(\xi)$, are presented in Fig.1, for various σ/A . These histograms are compared with corresponding Gaussian pdfs. The agreement is extremely high. For small noise the histograms are well separated, meaning that there will be no false detection of maxima. The IF estimation will be reliable. However, for higher noise values, the histograms intersect, meaning that there is a significant probability of false maxima position detection. The expected and obtained number of false detections [in 5120 simulation according to (4)], for various σ/A and $N = 256$, are given in Table I.

Approximation of P_E : For well separated histograms (for small σ/A) we can write $G(\xi)p(\xi) \approx MQ(\xi)p(\xi)$. Then it is easy to derive that:

$$P_E \approx \frac{M}{2\pi\sigma_{WD}^2} \times \int_{-\infty}^{\infty} \int_x^{\infty} e^{-y^2/2\sigma_{WD}^2} e^{-(x-A_{WD})^2/2\sigma_{WD}^2} dx dy = \frac{M}{2} \operatorname{erfc}\left(\frac{A_{WD}}{2\sigma_{WD}}\right). \quad (5)$$

Thus, the relevant parameter for this error is *the maximal AT to distribution standard deviation ratio*.

TABLE I
NUMBER OF FALSE DETECTIONS IN 5120 INSTANTS: I) EXPECTED NUMBER, II) OBTAINED NUMBER.

σ/A	P_E	I	II
0.2	7.4793e-224	0	0
0.4	2.8395e-038	0	0
0.6	7.9544e-014	0	0
0.8	3.9741e-006	0	0
1.0	4.7100e-003	24	20
1.2	9.0217e-002	469	442
1.4	3.2033e-001	1640	1687
1.6	5.6288e-001	2882	2921

A. Mean square error (MSE)

For the WD with N samples along the frequency axis, the true IF value $\hat{\omega}(t) = \omega(t)$ is obtained with probability $1 - P_E$, while other (false) values are detected with probability $P_E/(N - 1)$. Considering the frequencies $\omega_k = k\pi/N$, $-N/2 \leq k < N/2$, the mean estimation error for large number of samples N is $E\{\Delta\hat{\omega}(t)\} \approx P_E\omega(t)$. The MSE depends on $\omega(t)$. It is smallest for the IF at the middle point $\omega(t) = 0$, $E\{\Delta\hat{\omega}^2(t)\} \approx P_E\pi^2/12$. The largest MSE, $E\{\Delta\hat{\omega}^2(t)\} \approx P_E\pi^2/3$, is obtained for $\omega(t) = \pm\pi/2$. The mean value of MSE is of order $e_m \approx 2P_E$.

To compare this kind of error with the error due to the small IF deviations within the AT, note that its MSE is $e_s = 12\sigma^2(1 + \sigma^2/A^2)/(A^2N^3)$, [2]. This kind of error is always present when the AT is not completely concentrated at a single point along the frequency axis. The values of e_m and e_s for various N are presented in Fig.2. The errors are of different orders of magnitude, before and after their intersection point. Thus, in each region, only one of them could be considered as dominant.

III. GENERALIZATION

Consider a distribution from the CD class, whose kernel $c(\theta, \tau)$ is appropriately discretized. If we again assume that the values of the noisy distribution are Gaussian we can use (4). Distribution values may be considered

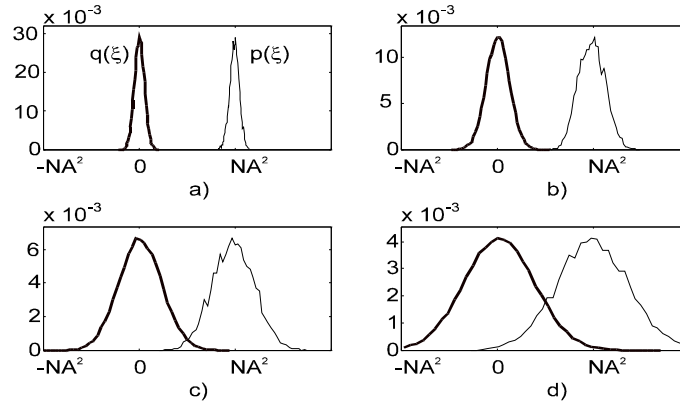


Fig. 1. Histograms of the WD values for a) $\sigma/A = 0.5$, b) $\sigma/A = 0.8$, c) $\sigma/A = 1.1$, and d) $\sigma/A = 1.4$. Thin lines: histograms of the WD values along the IF; thick lines: histograms of the WD values outside the AT.

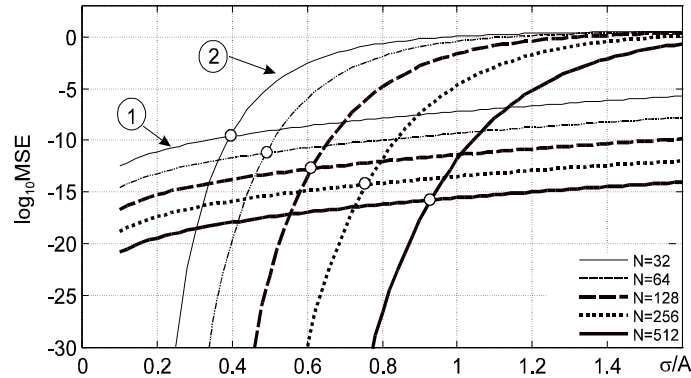


Fig. 2. MSE of IF estimation for various N : 1-MSE caused by the small variations of maxima within the auto-term, and 2-MSE caused by false maxima detection.

as $\mathcal{N}(A_{CD}, \sigma_{CD})$ or $\mathcal{N}(0, \sigma_{CD})$, where A_{CD} is the AT maximal value in the absence of noise, and the variance is approximated by [8]:

$$\sigma_{CD}^2 = 4\sigma^2(A^2 + \sigma^2) \sum_{\theta} \sum_{\tau} |c(\theta, \tau)|^2 / N.$$

As an example, consider the Choi-Williams distribution (CWD), $c(\theta, \tau) = \exp(-\theta^2 \tau^2 / \alpha^2)$, and the signal $f(n) = A \exp(jan^2/2)$. The AT maximum is $A_{CD} = A^2 \sum_{\tau=-N/2}^{N/2-1} c(-a\tau, \tau)$ [6]. In order to have a good estimation of P_E , the false estimation is indicated only when a value outside the AT is detected. The AT width is calculated in advance for a given a and parameter α [6]. Histograms of the CWD values, within and outside the AT, are given in Fig.3 for various α and a and for $\sigma = 0.8$. Table II presents the expected (E_{α}) and obtained

(O_{α}) number of false detections for various α , a , and σ , in 5120 instants. For $a = 0$, the error probability is smaller for narrow kernel ($\alpha = 1$) since σ_{CD}^2 is significantly reduced, while small α does not influence the AT form. For highly nonstationary signal ($a = \pi/N$), narrow kernel with $\alpha = 1$ significantly degrades the AT. The histogram of AT values is moved toward zero. By widening the kernel (toward the WD) Fig.3d-f, we improve histograms separation, expecting smaller errors (columns V-XII in Table II). Note that for narrow kernels, a significant IF estimation error within their wide AT exists, as well.

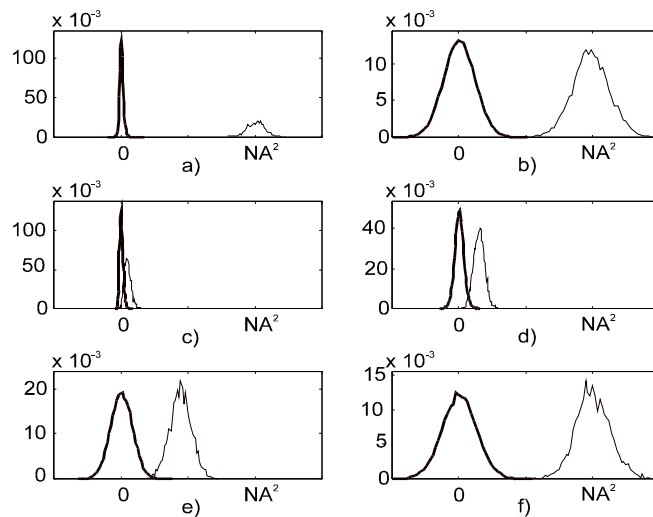


Fig. 3. Histogram of the CD values for $\sigma/A = 0.8$ with: a) $a = 0, \alpha = 1$, b) $a = 0, \alpha = 1000$, c) $a = \pi/N, \alpha = 1$, d) $a = \pi/N, \alpha = 10$, e) $a = \pi/N, \alpha = 100$, and f) $a = \pi/N, \alpha = 1000$. Thin lines: histograms of the CD values along the AT; thick lines: histograms of the CD values outside the AT.

TABLE II

FALSE DETECTIONS IN 5120 INSTANTS. E: EXPECTED NUMBER, O: OBTAINED NUMBER, INDEX α IS PARAMETER OF KERNEL FUNCTION.

σ	$a = 0$				$a = \pi/N$							
	CWD		\sim WD		$>$ From CWD toward WD \rightarrow							
	E ₁	O ₁	E ₁₀₀₀	O ₁₀₀₀	E ₁	O ₁	E ₁₀	O ₁₀	E ₁₀₀	O ₁₀₀	E ₁₀₀₀	O ₁₀₀₀
0.2	0	0	0	0	0	0	0	0	0	0	0	0
0.4	0	0	0	0	0	0	0	0	0	0	0	0
0.6	0	0	0	0	322	664	15	19	0	0	0	0
0.8	0	0	0	0	2109	2101	713	605	44	31	0	0
1.0	0	0	19	20	3655	3622	2426	2313	780	720	30	31
1.2	0	0	342	408	4399	4058	3712	3438	2288	2297	509	628
1.4	0	0	1546	1543	4726	4483	4371	3770	3504	3316	1720	1631
1.6	0	0	2535	2795	4879	4614	4688	4366	4202	3925	2952	2931

IV. CONCLUSION

The IF estimation based on the TF distributions of highly noisy signals is considered. It has been shown that the crucial analysis parameter is the AT magnitude to the distribution standard deviation ratio. There exists a critical point when the errors due to the variations within the AT and false maxima detection are the same. These two kind of errors are of different orders of magnitude, before and after this point, meaning that only one of them could be considered as dominant in each re-

gion.

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