Performance of Spectrogram as IF Estimator

LJubiša Stanković, Miloš Daković, Veselin N. Ivanović

Abstract— **Exact expressions for the variance and bias of the instantaneous frequency (IF) estimate using a spectrogram are derived. Simple approximative formulae are provided and theoretical results are statistically confirmed.**

I. Introduction and Problem Formulation

The spectrogram is still the most commonly used tool for time-frequency analysis. It is known that, when used as an instantaneous frequency (IF) estimator [1]-[3], the spectrogram produces unbiased estimates when the signal phase is a linear or quadratic function of time. The variance has been derived under the condition that the IF variations within the lag window are small [4]. The variance and bias of an IF estimate obtained using a spectrogram, for general frequency modulated signals, are derived. Special attention is devoted to signals whose IF variations, within the lag window, could be considered as linear functions of time. For this case, a very simple approximative formula for the variance is given. The rate of IF changes within the lag window increases the variance exponentially. The estimation is unbiased for linear IF changes, while the bias can be approximated by a linear function for quadratic IF variations. The derived expressions are checked statistically.

Consider discrete-time noisy signal observations:

$$x(nT) = f(nT) + \varepsilon(nT) \tag{1}$$

of the continuous signal $f(t) = A(t)\exp(j\phi(t))$ with slow-varying amplitude with respect to the phase function $\phi(t)$, where T is a sampling interval and $\varepsilon(nT)$ is white, Gaussian i.i.d. noise. The spectrogram of this signal can be written as:

$$S_x(t,\omega;w_h(nT))$$

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$$= \left| \sum_{n=-\infty}^{\infty} w_h(nT)x(t+nT)e^{-j\omega nT} \right|^2 \tag{2}$$

where $w_h(nT) = w(nT/h)T/h$ and $w(\tau)$ is the real-valued, symmetric, finite length window function. Parameter $h > 0$ denotes the length of the window, and localizes the IF estimate.

By definition, the IF of the considered signal is $\omega(t) \equiv d\phi(t)/dt$. The value $\omega(t)$ is usually estimated as

$$\hat{\omega}_h(t) = \arg[\max_{\omega \in Q_w} S_x(t,\omega;w_h(\tau))].$$

The estimation error is defined as $\Delta\hat{\omega}_h(t) = \omega(t) - \hat{\omega}_h(t)$. Owing to the presence of noise $\varepsilon(nT)$, the estimation error $\Delta\hat{\omega}_h(t)$ can be considered as a random variable, characterized by its bias and variance.

II. Analysis

The IF is located at the stationary points of $S_x(t,\omega;w_h(nT))$. To perform estimation error analysis, we linearize $\partial S_x(t,\omega;w_h(nT))/\partial\omega$ around the stationary point with respect to the small estimation error $\Delta\hat{\omega}_h(t)$, phase residue $\triangle\phi$ (third- and higher-order terms in a Taylor expansion of phase $\phi(t)$), and noise ε:

$$\frac{\partial S_x(t,\omega;w_h(nT))}{\partial\omega}\Big|_0 + \frac{\partial^2 S_x(t,\omega;w_h(nT))}{\partial\omega^2}\Big|_0 \Delta\hat{\omega}_h(t)$$
$$+\frac{\partial S_x(t,\omega;w_h(nT))}{\partial\omega}\Big|_0 \delta_{\triangle\phi} + \frac{\partial S_x(t,\omega;w_h(nT))}{\partial\omega}\Big|_0 \delta_\varepsilon = 0 \tag{3}$$

where $|_0$ indicates that the derivatives are calculated at the point $\omega = \phi'(t)$, $\triangle\phi = 0$, and $\varepsilon = 0$. Details can be found in [5]. Note that the first term in eq.(3) has zero value at this point. In the second term, we have a deterministic expression:

$$D_f = \frac{\partial^2 S_x(t,\omega;w_h(nT))}{\partial\omega^2}\Big|_0$$
$$= -2[\mathrm{Re}\{F_f(t,\phi'(t);(nT)^2 w_h(nT))\times$$

$$F_f^*(t,\phi'(t);w_h(nT))\} - S_f(t,\phi'(t);nTw_h(nT))] \quad (4)$$

while the third and the forth terms influence only the IF estimation bias and variance, respectively. From eq.(3), we can easily obtained the expression for $\Delta\hat{\omega}_h(t)$. This will be used for the estimation error analysis.

Variance: Since $\triangle\phi$ is not a random variable, it does not influence the estimation variance, thus from eq.(3) we have

$$\text{var}\{\Delta\hat{\omega}_h(t)\} = \text{var}\left\{\frac{\partial S_x(t,\omega;w_h(nT))}{\partial\omega}|_0\delta_\epsilon\right\}/D_f^2. \quad (5)$$

By simple calculation, we can evaluate the numerator expression in eq.(5), resulting in

$$\text{var}\{\Delta\hat{\omega}_h(t)\} = 2\sigma_\varepsilon^2[S_f(t,\phi'(t);nTw_h(nT))\frac{T}{h}M_0^{w^2} + S_f(t,\phi'(t);w_h(nT))ThM_2^{w^2}]/D_f^2 \quad (6)$$

where $\text{Re}\{\cdot\}$ denotes a real part, $F_f(t,\omega, w_h(nT))$ denotes the discrete-time short-time Fourier transformation of the signal $f(t)$,

$$F_f(t,\omega;w_h(nT)) = \sum_{n=-\infty}^{\infty} w_h(nT)f(t+nT)e^{-j\omega nT}.$$

Higher powers of the total noise variance σ_ε^2 are disregarded, and the fact that the window $w_h(\tau)$ is real and symmetric is used. In eq.(6), M_k^w denotes the kth moment of the window, $M_k^w \simeq \int_{-\infty}^{\infty} w(\tau)(\tau)^k d\tau$. The moments $M_k^{w^2}$ are calculated for the squared window $w^2(\tau)$.

Variance for linear FM signal: Consider a linear FM signal with constant amplitude $A(t) = A$, and phase $\phi(t) = at^2/2$. Eq.(6) becomes

$$\text{var}\{\Delta\hat{\omega}_h(t)\} = \sigma_\varepsilon^2 \times \left|\sum_{n=-\infty}^{\infty} w_h(nT)e^{ja(nT)^2/2}\right|^2 \frac{ThM_2^{w^2}}{2A^2D_a^2} \quad (7)$$

where

$$D_a = \text{Re}\{\sum_{n=-\infty}^{\infty}(nT)^2 w_h(nT)e^{ja(nT)^2/2} \times \sum_{n=-\infty}^{\infty} w_h(nT)e^{-ja(nT)^2/2}\}.$$

Parameter a appears in the exponent in eq.(7), so we can assume that an approximative formula for eq.(7) can be found as $\sigma_\varepsilon^2\exp(P(a))$, where $P(a)$ is a polynomial in a. Coefficients of the polynomial $P(a)$ can be obtained by expanding $\ln(\text{var}\{\Delta\hat{\omega}_h(t)\})$ into a Taylor series around $a = 0$. Since $P(a)$ contains even powers of a only, if we take its first two terms we obtain

$$\text{var}\{\Delta\hat{\omega}_h(t)\} \simeq \frac{\sigma_\varepsilon^2 T}{2|A|^2h^3}\frac{M_2^{w^2}}{(M_2^w)^2}\exp(C_w a^2h^4) \quad (8)$$

where

$$C_w = \frac{1}{4}[(M_2^w/M_0^w)^2 + M_6^w/M_2^w - 2M_4^w/M_0^w]$$

is a window $w(\tau)$ dependent parameter. The relative error of approximation in eq.(8) is $1 - \exp(O(a^4h^8))$, where $O(\cdot)$ is the Landau symbol.

Eq.(8) is an approximation of eq.(7) for small values of a. An approximative formula for large values of a can be obtained by applying the stationary phase method [1] in eq.(6). Using the facts that

$$F_f(t,\omega;w(\tau)) \simeq Aw(\frac{\omega-at}{ah})\exp(j(\omega t-\frac{\omega^2}{2a}))\sqrt{2\pi j/a},$$

and

$$\partial^2 F_f(t,\omega;w(\tau))/\partial\omega^2 = -F_f(t,\omega;\tau^2 w(\tau)),$$

we have

$$\text{var}\{\Delta\hat{\omega}_h(t)\} \simeq \frac{\sigma_\varepsilon^2}{|A|^2}\frac{M_2^{w^2}}{4\pi(w^{(2)}(0))^2}Th^7a^5 \quad (9)$$

where $w^{(2)}(0)$ is the second derivative of the window function $w(\tau)$ at $\tau = 0$.

Bias: The bias of the estimator is the expected value of the error $\Delta\hat{\omega}_h(t)$. By evaluating the third term in eq.(3), we obtain:

$$\text{bias}(\Delta\hat{\omega}_h(t)) = -2\,\text{Im}\{F_f(t,\phi'(t);nTw_h(nT)) \times F_f^*(t,\phi'(t);w_h(nT))\}/D_f. \quad (10)$$

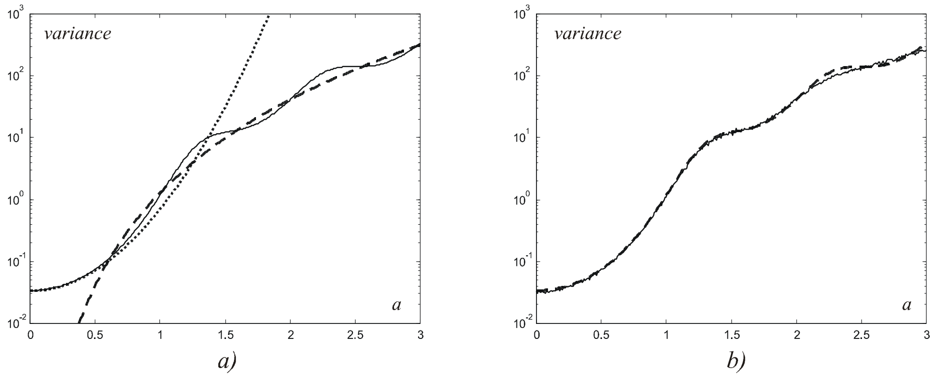

Fig. 1. IF variance obtained: a) theoretically by eq.(7) (solid line), and approximately by eqs. (8) (dotted line), and (9) (dashed line); b) theoretically (dashed line), and statistically (solid line). Values of the IF direction a, normalized with the maximal frequency $\omega_m = \pi/T$, are shown in the x-axis.

In the case of constant and linear FM signals, the bias is equal to zero. For signals with quadratic IF $f(t) = A\exp(jbt^3/6)$, eq.(10) can be expanded into a Taylor series around $b = 0$, and we get the linear function of b, $\text{bias}(\Delta\hat{\omega}_h(t)) \cong b(h^2 M_4^w/(6M_2^w)$. The approximation error is $O(h^8 b^3) + O(t^2 h^6 b^3)$.

III. Example and Conclusions

The signal $f(t) = \exp(jat^2/2)$ is considered for various values of parameter a. The sampling interval is $T = 1/256$, and 64 point Hanning window is used ($h = 1/4$). The total noise $\varepsilon(nT)$ variance is $\sigma_\varepsilon = 0.1$. Spectrograms are calculated in 256 points. The variance is time independent in this case, so the time instant $t = 0$ is considered.

Fig. 1a) shows variances obtained from eq.(7) (solid line), and from approximative formulae in eq.(8) (dotted line) and eq.(9) (dashed line). For small values of a eq.(8) gives appropriate results, while for large a eq.(9) gives good approximation. The highest value of a for which eq.(8) can be used is $C_w a^2 h^4 < 5$. This value is a lower limit for eq.(9). For very small $C_w a^2 h^4 < 1/5$, narrow windows or small a, the variance can be treated as a constant [4]. The values of a, normalized with the maximal frequency for the sampling interval T, are shown on the x-axis $a_n = a/\omega_m = aT/\pi$. Fig. 1b) shows theoretically and statistically obtained variances. The statistically obtained variance is calculated as a mean value of 2048 different realizations. The agreement between theoretically and statistically obtained data is very high.

Note that the variance values in Fig. 1 are calculated for a Hanning window with 64 points. If we used 32 points with the same sampling interval, i.e. $h = 1/8$, then the x-axis would be rescaled by a factor of $1/2^2$, meaning that we can assume that the variance is constant up to the normalized $a = 4/5 \simeq 1$, and that eq.(8) can be used up to $a = 20$. Of course, a wider window means axis rescaling in the opposite direction. For the analyzed signal, the bias is zero.

IV. Acknowledgment

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References

[1] B. Boashash: "Estimating and interpreting the instantaneous frequency of a signal - Part 1: Fundamentals", *Proc. IEEE*, vol.80, Apr.1992, pp.519-538.

[2] P. Rao, F.J. Taylor: "Estimation of the instantaneous frequency using the discrete Wigner distribution", *Electronics Letters*, vol.26, 1990, pp.246-248.

[3] D.C.Reid, A.M.Zoubir, B.Boashash: "Aircraft flight parameter estimation based on passive acoustic techniques using polynomial Wigner-Ville distribution", *J.Ac.Soc.Am.*, vol.102, no.1, 1997, pp.207-223.

[4] V. Katkovnik, LJ. Stanković: "Instantaneous frequency estimation using the periodogram with

time-varying window length", *Signal Processing*, vol.67, no.3, June 1998, pp.345-358.

[5] V. Katkovnik, LJ. Stanković: "Instantaneous frequency estimation using the Wigner distribution with varying and data-driven window length", *IEEE Trans. on Signal Processing*, vol.46, no.9, Sept.1998, pp.2315-2326.