

Parametric Estimation of the FM Signals Using Wigner Distribution-based Maximum Likelihood Estimator

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Abstract— Parametric estimation of the monocomponent noisy signals is considered in this paper. The Wigner distribution-based maximum likelihood (ML) estimator is used as estimation tool. Various types of the FM signals and noisy environments are considered.

I. INTRODUCTION

The Wigner distribution (WD), after 70 years of its introduction in quantum mechanics [1], and after 55 years of the first application in the field of spectral analysis [2], remains a very useful tool for the time-frequency (TF) signal analysis. We want to mention the recent book [3] as a comprehensive overview of the WD applications. The instantaneous frequency (IF) estimation is one of the most important applications of the WD [4]. It can be used in both parametric and non-parametric methods for the IF estimation [4]-[8]. Some of the non-parametric methods for the IF estimation are described in [4], [5]. Combinations of the WD with Radon or Hough transformation are the common tool for parametric IF estimation [6]-[8]. These methods can be considered as projections of the TF values along assumed IF lines. In this paper we consider the parametric IF estimation by using the WD-based maximum likelihood (ML) estimator. This estimator has been applied in [9] for determination of the signal position and duration. The IF estimation in various noisy environments is considered. Estimation of signal parameters is done based on the maximum of the proposed sub-optimal log-likelihood function.

The paper is organized as follows. An SICE Annual Conference, SICE'2002, Chiba, Japan.

overview of the WD behavior in noisy environment and the ML estimator are discussed in Section II. Determination of the WD auto-covariance matrix in various noisy environments is presented in Section III. Algorithm for the IF estimation is given in Section IV. Numerical examples are presented in Section V.

II. WIGNER DISTRIBUTION OF NOISY SIGNALS

Consider a FM signal $f(t) = A \exp(j\phi(t; \mathbf{a}))$, where \mathbf{a} is a set of signal parameters that should be estimated, embedded in the white noise $\nu(t)$ with variance σ_ν^2 :

$$x(t) = f(t) + \nu(t) = A \exp(j\phi(t; \mathbf{a})) + \nu(t). \quad (1)$$

The signal sampling rate is Δt , $x(n) = x(n\Delta t)$. The WD can be represented in the following form:

$$WD_{xx}(n, k) = \sum_{m=0}^{N-1} x(n+m)x^*(n-m)e^{-j4\pi mk/N}. \quad (2)$$

Mean value of the WD is $E\{WD_{xx}(n, k)\} = WD_{ff}(n, k) + \sigma_\nu^2$, and the auto-covariance is

$$\begin{aligned} R(n_1, k_1, n_2, k_2) &= \\ &E\{WD_{xx}(n_1, k_1)WD_{xx}^*(n_2, k_2)\} \\ &- E\{WD_{xx}(n_1, k_1)\}E\{WD_{xx}^*(n_2, k_2)\} \\ &= 2\sigma_\nu^2 \cos(2\pi k_2(n_2 - n_1)/N) \\ &\times \text{Re} \left\{ \sum_{m=0}^{N-1} f(n_1 + m) \times \right. \end{aligned}$$

$$\begin{aligned}
& f^*(2n_2 - n_1 + m)e^{-\frac{j4\pi(k_1 - k_2)m}{N}} \Big\} \\
& + \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} E\{\nu(n_1 + m_1)\nu^*(n_1 - m_1) \\
& \quad \times \nu^*(n_2 + m_2)\nu(n_2 - m_2)\} \\
& \times e^{-j(4\pi k_1 m_1 - 4\pi k_2 m_2)/N} - \sigma_\nu^4. \quad (3)
\end{aligned}$$

The sum used for calculation of the WD (2) usually contains a large number of terms $x(n+m)x^*(n-m)e^{-j4\pi mk/N}$ ($N = 128$, $N = 256$, ...). Therefore, the central limit theorem may be applied to the WD and its values can be treated asymptotically as the Gaussian random field. Thus, the joint pdf of the WD values can be written as:

$$\begin{aligned}
p\{\mathbf{w}\} &= \frac{|\mathbf{R}|^{-\frac{1}{2}}}{(2\pi)^{N^2/2}} \times \\
& \exp\{-\frac{1}{2}(\mathbf{w} - \mathbf{m})^T \mathbf{R}^{-1}(\mathbf{w} - \mathbf{m})\}, \quad (4)
\end{aligned}$$

where \mathbf{w} is the vector consisting of WD values, $\mathbf{w} = [W_{00}, W_{01}, \dots, W_{N-1, N-1}]^T$, $W_{ij} = WD_{xx}(n_i, k_j)$, $n_i, k_j \in [0, N)$, $\mathbf{m} = [E\{W_{ij}\}]$ and \mathbf{R} is the auto-covariance matrix $\mathbf{R} = [r_{ij}]$ where $r_{ij} = E\{[\mathbf{w}(i) - \mathbf{m}(i)][\mathbf{w}(j) - \mathbf{m}(j)]\}$, $\mathbf{w}(i)$ and $\mathbf{m}(i)$ represent the i th elements of the vectors \mathbf{w} and \mathbf{m} , respectively. The WD-based ML estimator is applied in [9] for the estimation of position and duration of signal. Two hypotheses are introduced in order to test if the $x(n)$ contains signal $f(n)$ with the parameter vector \mathbf{a} :

H^1 : $WD_{xx}(n, k; \mathbf{a})$ - signal and noise case,

H^0 : $WD_{\nu\nu}(n, k)$ - noise only case. (5)

Estimation of the signal parameters can be done by using the log-likelihood ratio:

$$L(\mathbf{a}) = \ln p\{\mathbf{w}|H^1\}/p\{\mathbf{w}|H^0\}, \quad (6)$$

where the likelihood function $p\{\mathbf{w}|H^i\}$ is given as:

$$\begin{aligned}
p\{\mathbf{w}|H^i\} &= \frac{|\mathbf{R}_i|^{-1/2}}{(2\pi)^{N^2/2}} \\
& \times \exp\{-\frac{1}{2}(\mathbf{w} - \mathbf{m}_i)^T \mathbf{R}_i^{-1}(\mathbf{w} - \mathbf{m}_i)\} \quad (i = 0, 1) \quad (7)
\end{aligned}$$

and \mathbf{m}_i and \mathbf{R}_i are the mean value and the auto-covariance matrix under H^i hypothesis. Since the auto-covariance matrix is of large dimension $N^2 \times N^2$, the determination of this matrix and its inverse can be very time-consuming process. Therefore, it will be rather preferable to take a faster procedure for determination of the log-likelihood ratio. At this stage, it should be noted that the maximization of the likelihood ratio $L(\mathbf{a})$ with respect to the unknown vector \mathbf{a} is equivalent to the maximization of $\ln p\{\mathbf{w}|H^1\}$ since $p\{\mathbf{w}|H^0\}$ does not contain any component of the parameter vector \mathbf{a} . Instead of $L(\mathbf{a})$, given with (6), the following form of the log-likelihood ratio will be used:

$$\tilde{L}(\mathbf{a}) = \sum_{n=0}^{N-1} \ln p\{\mathbf{w}_n|H^1\}/p\{\mathbf{w}_n|H^0\}, \quad (8)$$

where \mathbf{w}_n represents the WD values at the instant n , $\mathbf{w}_n = [W_{n0}, W_{n1}, \dots, W_{n, N-1}]^T$, $n \in [0, N)$, with:

$$\begin{aligned}
p\{\mathbf{w}_n|H^i\} &= \frac{|\mathbf{R}_{n,i}|^{-1/2}}{(2\pi)^{N/2}} \times \exp\{-\frac{1}{2}(\mathbf{w}_n - \mathbf{m}_{n,i})^T \\
& \quad \mathbf{R}_{n,i}^{-1}(\mathbf{w}_n - \mathbf{m}_{n,i})\} \quad (i = 0, 1), \quad (9)
\end{aligned}$$

while $\mathbf{m}_{n,i}$ and $\mathbf{R}_{n,i}$ are the vector of the mean values and the auto-covariance matrix of the WD values at instant n under H^i . The auto-covariance matrix $\mathbf{R}_{n,i}$ has dimensions $N \times N$, and the demands for its inverse-matrix calculation are reasonable.

III. AUTO-COVARIANCE MATRIX

An overview of the auto-covariance matrix determination is given in this section for various noisy environments. The simplest cases are presented first, followed by the more complex ones.

A. Complex Gaussian Noise

The auto-covariance matrix \mathbf{R}_n follows from (3) for $n_1 = n_2 = n$. For a signal with constant amplitude A embedded in a complex Gaussian noise it is given as [10]:

$$R(k_1, k_2) = N\sigma_\nu^2(2A^2 + \sigma_\nu^2)\delta(k_1 - k_2). \quad (10)$$

In this case the auto-covariance matrix is diagonal with well-known expression for variance of the WD [10], [11]. Its determinant is given as $|\mathbf{R}_{n,i}| = [N\sigma_\nu^2(2A^2 \cdot i + \sigma_\nu^2)]^N$, while the inverse $\mathbf{R}_{n,i}^{-1}$ is diagonal with $1/N\sigma_\nu^2(2A^2 \cdot i + \sigma_\nu^2)$ values. The value $\Delta_{n,i} = (\mathbf{w}_n - \mathbf{m}_{n,i})^T \mathbf{R}_{n,i}^{-1} (\mathbf{w}_n - \mathbf{m}_{n,i})$ can be calculated as:

$$\Delta_{n,i} = \sum_{k=0}^{N-1} \frac{[WD_{xx}(n, k) - i \cdot WD_{ff}(n, k) - \sigma_\nu^2]^2}{N\sigma_\nu^2(2A^2 \cdot i + \sigma_\nu^2)}. \quad (11)$$

B. Complex Noise

In the case of complex noise with independent real and imaginary parts the auto-covariance matrix has the following form:

$$R(k_1, k_2) = \alpha_i \delta(k_1 - k_2) + \beta, \quad (12)$$

where $\alpha_i = N\sigma_\nu^2(2A^2 \cdot i + \sigma_\nu^2)$, and $\beta = E\{|\nu(n)|^4\} - 2\sigma_\nu^4$. In this case the auto-covariance matrix has value $[\alpha_i + \beta]$ on its main diagonal and β elsewhere. Its inverse matrix has the same form as $R(k_1, k_2)$ given by (12):

$$\mathbf{R}_{n,i}^{-1} = (a_i - b_i)\delta(k_1 - k_2) + b_i \quad (13)$$

with $a_i = [\alpha_i + (N-1)\beta]/[\alpha_i^2 + \alpha_i\beta(N+1)]$ and $b_i = -\beta/[\alpha_i^2 + \alpha_i\beta(N+1)]$. The determinant is $|\mathbf{R}_{n,i}| = \alpha_i^N + N\alpha_i^{N-1}\beta$. Calculation of $\Delta_{n,i}$ can be done as:

$$\Delta_{n,i} = (a_i - b_i) \times \sum_{k=0}^{N-1} [WD_{xx}(n, k) - i \cdot WD_{ff}(n, k) - \sigma_\nu^2]^2 + b_i \left[\sum_{k=0}^{N-1} [WD_{xx}(n, k) - i \cdot WD_{ff}(n, k) - \sigma_\nu^2]^2 \right]. \quad (14)$$

C. Real Gaussian Noise

In the case of the real Gaussian noise the auto-covariance matrix is:

$$R(k_1, k_2) = \alpha_i \delta(k_1 - k_2) + \gamma \delta(k_1 + k_2) + F(k_1, k_2), \quad (15)$$

where

$$F(k_1, k_2) = \sum_{m=0}^{N-1} f^2(n+m) e^{-j4\pi m(k_1+k_2)/N}. \quad (16)$$

To simplify our procedure we will use $\tilde{R}(k_1, k_2) = \alpha_i \delta(k_1 - k_2) + \gamma \delta(k_1 + k_2)$, neglecting the term $F(k_1, k_2)$. Thus, the matrix $\tilde{R}(k_1, k_2)$ has α_i along the main diagonal and $\gamma = N\sigma_\nu^4$ along the anti-diagonal. Calculation of $\mathbf{R}_{n,i}^{-1}$ under H_1 and H_0 is different. The inverse-matrix of the (15) under H_1 is of the same shape as (15) with $a = [2A^2 + \sigma_\nu^2]/[4A^2N(A^2 + \sigma_\nu^2)]$ at the main diagonal and $b = -1/[4A^2N(A^2 + \sigma_\nu^2)]$ along the anti-diagonal. The determinant of $\mathbf{R}_{n,1}$ is $|\mathbf{R}_{n,1}| = \sum_{l=0}^{N/2} \binom{N/2}{l} a^{2l} b^{N-2l}$. So, the value $\Delta_{n,1}$ can be obtained as:

$$\Delta_{n,1} = a \sum_{k=0}^{N-1} [WD_{xx}(n, k) - WD_{ff}(n, k) - \sigma_\nu^2]^2 + b \sum_{k=0}^{N-1} [WD_{xx}(n, k) - WD_{ff}(n, k) - \sigma_\nu^2] \times [WD_{xx}(n, N-k-1) - WD_{ff}(n, N-k-1) - \sigma_\nu^2]. \quad (17)$$

For H^0 , it is assumed that there is no signal and that only Gaussian noise is received. The auto-covariance matrix of the real signal is symmetric: $R(k_1, k_2) = R(k_1, N-1-k_2) = R(N-1-k_1, k_2) = R(N-1-k_1, N-1-k_2)$. Thus, instead of the entire matrix $R(k_1, k_2)$, under H^0 one uses its sub-matrix $\mathbf{\Gamma}$ of $N/2 \times N/2$ -dimension with $k_1 < N/2$ and $k_2 < N/2$. Then, $p\{\mathbf{w}_n|H^0\}$ can be calculated as $p\{\mathbf{w}_n|H^0\} = p^2$, where

$$p = \frac{|\mathbf{\Gamma}|^{-\frac{1}{2}}}{(2\pi)^{\frac{N}{4}}} \times$$

$$\exp\left\{-\frac{1}{2}(\mathbf{w}_n - \mathbf{m}_{n,0})^T \mathbf{\Gamma}^{-1}(\mathbf{w}_n - \mathbf{m}_{n,0})\right\}. \quad (18)$$

Vectors \mathbf{w}_n and $\mathbf{m}_{n,0}$ are reduced on their parts with $k < N/2$. Since $\mathbf{\Gamma}$ has diagonal shape, the calculation of $\mathbf{\Gamma}^{-1}$ can be obtained similarly as in the case of the complex Gaussian noise.

D. Real Noise

In this case the matrix $\tilde{R}(k_1, k_2)$ has a form:

$$\tilde{R}(k_1, k_2) = \alpha_i \delta(k_1 - k_2) + \gamma \delta(k_1 + k_2) + \beta, \quad (19)$$

where $\beta = E\{\nu(n)^4\} - 3\sigma_\nu^4$. Under H^1 the inverse matrix has the same shape as (19):

$$\mathbf{R}_{n,1}^{-1} = (a - c)\delta(k_1 - k_2) + (b - c)\delta(k_1 + k_2) + c, \quad (20)$$

where:

$$a = \frac{\alpha_1^2 + 2\beta^2 + 2\alpha_1\beta + (N-1)\alpha_1\gamma + N\gamma\beta}{\alpha_1(\alpha_1 + 2\beta)(\alpha_1 + N\beta + 2\gamma)}, \quad (21)$$

$$b = -\frac{\gamma}{(\alpha_1 + 2\beta)(\alpha_1 + N\beta + 2\gamma)}, \quad (22)$$

$$c = -\frac{\alpha_1^2 + 2\beta^2 + 2\alpha_1\beta + 3\alpha_1\gamma + \gamma^2(N-1)}{\alpha_1(\alpha_1 + 2\beta)(\alpha_1 + N\beta + 2\gamma)}. \quad (23)$$

Similar procedure as in the case of the real Gaussian noise can be performed for calculation of $p\{\mathbf{w}_n|H^0\}$. The only difference is in the shape $\mathbf{\Gamma}$. Namely, its shape is now as in the complex noise case stated in Subsection III.B.

E. Real Signal and Real Noise

This is a very realistic case. It can be fully reduced to the case of the complex signal embedded in the complex noise (Subsection III.B). The only difference is that instead of the $\mathbf{R}_{n,i}$ one should use the sub-matrix for $k_1 < N/2$ and $k_2 < N/2$, and that instead of $\mathbf{w}_{n,i}$ and $\mathbf{m}_{n,i}$ one should use the corresponding sub-vectors. Therefore, the calculation demands are reduced.

F. Other TF Representation

In the case of other TF representations, belonging to the Cohen class of distributions, noise is strongly signal-dependent. Therefore, the calculation of auto-covariance matrix and vector of the mean values should be performed for each instant [12]. This fact increases calculation demands of the algorithm and again shows that the WD possesses some very important advantages with respect to the other TF distributions.

IV. ALGORITHM

Algorithm for the estimation of the FM signal's parameters can be summarized as follows:

1. Calculation of the $WD_{xx}(n, k)$.
2. Estimation of the noise variance and signal amplitude. The noise variance can be estimated as:

$$\hat{\sigma}_\nu^2 = \frac{1}{0.6745\sqrt{2}}[R^2 + jI^2], \quad (24)$$

$$R = \text{median}\{|\text{real}(x(n) - x(n-1))| | n \in [1, N]\} \quad (25)$$

$$I = \text{median}\{|\text{imag}(x(n) - x(n-1))| | n \in [1, N]\} \quad (26)$$

while the amplitude estimation is: $\hat{A}^2 = \frac{1}{N} \sum_n |x(n)|^2 - \hat{\sigma}_\nu^2$.

3. Under H^0 the mean value of the WD is $E\{WD_{xx}(n, \omega)\} = \sigma_\nu^2$, and estimation of $\mathbf{m}_{n,0}$ is given as $\mathbf{m}_0 = \hat{\sigma}_\nu^2$. Estimation of the auto-covariance matrix $\mathbf{R}_{n,0}$ depends on the considered noise case.

4. Under H^1 we have $\mathbf{m}_{n,1} = WD_{ff}(n, \omega; \mathbf{a}) + \hat{\sigma}_\nu^2$, while calculation of $\mathbf{R}_{n,1}$ depends on noise case.

5. Determination of log-likelihood ratio $\tilde{L}(\mathbf{a})$ for each \mathbf{a} from the considered set.

6. Maximum of $\tilde{L}(\mathbf{a})$ determines the IF parameters [9]:

$$\hat{\mathbf{a}} = \arg \left\{ \max_{\mathbf{a}} \tilde{L}(\mathbf{a}) \right\}. \quad (27)$$

V. EXAMPLES

Example 1: Consider a linear FM signal: $f(t) = A \exp(jat^2/2 + jbt)$, with parameter set $\mathbf{a} = (a, b) = (32\pi, 16\pi)$. Signal is considered within $t \in [-1, 1]$, with the sampling rate $\Delta t = 1/128$. The WD of noise-free signal is shown in Figure 1a. Signal is embedded in a white complex Gaussian noise with $SNR = -10$ [dB]. The WD of noisy signal is shown in Figure 1b. Estimation is performed with proposed algorithm, where the set of the considered parameters was $\mathbf{L}_{ab} = \{[-64\pi, 64\pi] \times [-64\pi, 64\pi]\}$. Log-likelihood ratio $\tilde{L}(\mathbf{a})$ is shown in Figure 1c. From this figure one can readily see the estimate of the signal parameters.

Example 2: Consider a sinusoidal FM signal: $f(t) = A \exp(ja \sin(\omega_0 t/a))$, with parameters $\mathbf{a} = (\omega_0, a) = (16\pi, 16\pi)$. Signal is

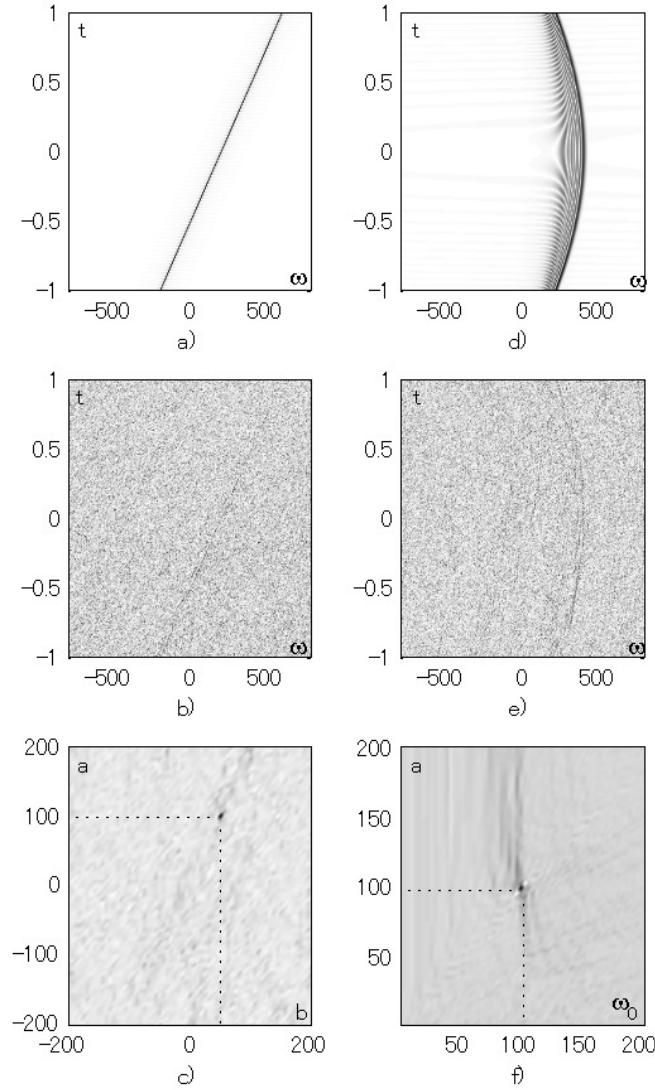


Fig. 1. Parametric estimation of the linear FM signal corrupted with complex Gaussian noise (left column) and sinusoidal FM signal embedded in complex uniform noise (right column): a), d) WDs of noise-free signal; b), e) WDs of noisy signal; c), f) $\hat{\mathbf{L}}(\mathbf{a})$.

considered within the same time-interval and with the same sampling rate as in the previous example. The WD of this signal is shown in Figure 1d. Signal is corrupted with the white uniform complex noise with $SNR = -10$ [dB]. The WD of the noisy signal is shown in Figure 1e. Signal parameters are estimated within $\mathbf{L}_{\omega_0 a} = \{[0, 32\pi] \times [0, 32\pi]\}$. Procedure for estimation of the auto-correlation matrix from

Section III.B is used. The value $E\{|\nu(n)|^4\}$ is estimated as $E\{|\nu(n)|^4\} \approx -9\hat{\sigma}_\nu^4/5$ by using the property of the uniform noise. Estimation of the signal parameters is shown in Figure 1f.

Example 3: Signal with parabolic IF is considered: $f(t) = A \exp(jat^3/3 + jbt^2/2 + jct)$, where $\mathbf{a} = (a, b, c) = (32\pi, 0, 16\pi)$. In order to simplify the visual presentation, the search is performed over parameter subspace

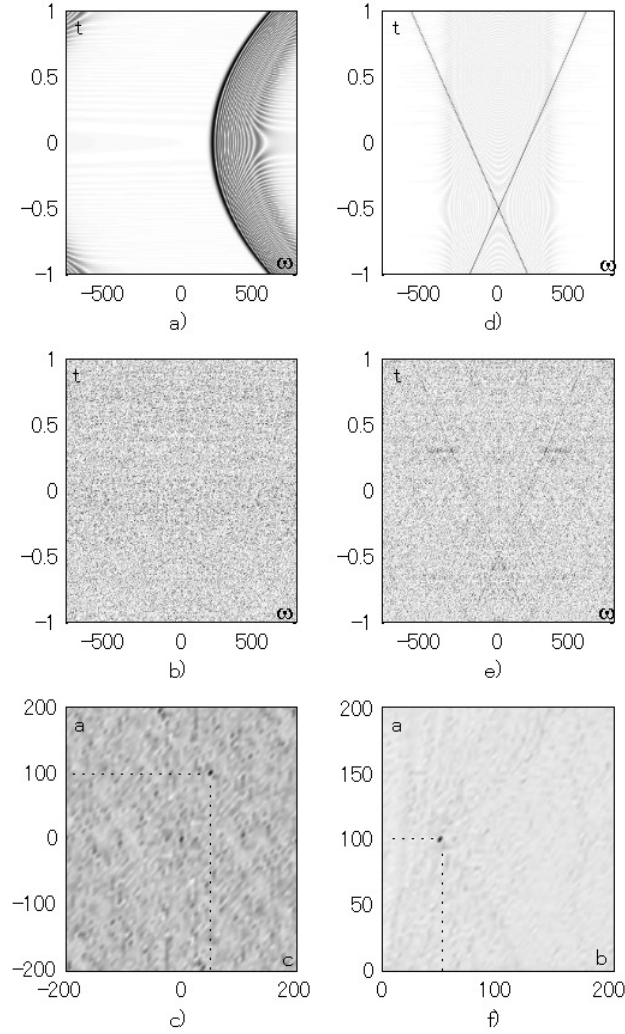


Fig. 2. Parametric estimation of the parabolic FM signal embedded in real Gaussian noise (left column), and real linear FM signal embedded in real Gaussian noise (right column): a), d) WDs of noise-free signal; b), e) WDs of noisy signal; c), f) $\tilde{L}(\mathbf{a})$.

for $b = 0$, $\mathbf{L}_{ac} = \{[-64\pi, 64\pi] \times [-64\pi, 64\pi]\}$. Signal is corrupted by a real Gaussian noise with $SNR = -10[\text{dB}]$. The determination of signal's parameters is performed according to Subsection III.C and Section IV. The WD of the noise-free signal is shown in Figure 2a, while the WD of noisy signal is shown in Figure 2b. Log-likelihood function $\tilde{L}(\mathbf{a})$ is shown in Figure 2c.

Example 4: Real linear FM signal: $f(t) = A \cos(at^2/2 + bt)$ is considered. Signal para-

eters are the same as in Example 1 with $SNR = -5[\text{dB}]$. Due to spectral symmetry, the search is performed in the modified parameter space $\mathbf{L}_{ab} = \{[0, 64\pi] \times [0, 64\pi]\}$. Corresponding WDs and log-likelihood function are shown in Figures 2d-2f.

VI. CONCLUSION

The ML estimator for the IF estimation is proposed in this paper. The WD is used as a tool for the IF estimation. Algorithm for

the IF estimation is presented. Determination of the algorithm parameters in various noisy environments is discussed. An advantage of the WD application, compared with other TF representation from the Cohen class, is noted. Numerical examples confirm our theoretical analysis.

ACKNOWLEDGMENTS

The work of I. Djurović is supported by the Postdoctoral Fellowship for Foreign Researchers of Japan Society for the Promotion of Science. Part of this research has been done by the support of the Ministry of Education, Culture, Sports, Science and Technology under Grant (C) 13650069 awarded to A. Ohsumi and Grant No.01215 awarded to I. Djurović.

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