

Robust Two-dimensional DFT

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Abstract— A form of the two-dimensional (2D) DFT, robust with respect to the influence of impulse noise, is defined. Different forms of the robust 2D DFT based on the iterative procedures and various median filter forms are introduced. The realization based on the robust 1D DFT along one coordinate and standard 1D DFT along the other one is also proposed. It performs similarly to other robust 2D DFT forms with significant calculation savings.

I. INTRODUCTION

Recently, the Huber robust statistic theory has found an application in the spectral analysis of signals corrupted with impulse noise [1]-[12]. Two forms of the robust DFT were introduced: 1) The robust M -DFT obtained by using the iterative procedure [3], [4]; and 2) The robust DFT based on the marginal median filter [6], [12]. Accuracy of both of them is similar. Namely, in the Gaussian noise environment the standard DFT performs slightly better than each of them. However, the standard DFT in the impulse noise environment produces poor results, while the robust DFT forms produce very high accuracy.

The spectral analysis of the 2D signals in impulse noise environment is considered in this paper. The robust 2D DFT is introduced in order to produce an accurate estimate of the non-noisy 2D DFT for signals corrupted by the impulse noise. Several forms of the robust 2D DFT are analyzed. They are based on the iterative procedure and various median filter forms.

The paper is organized as follows. As a brief introduction to the topic, filtering of signals by using the moving average and median filter, and the robust DFT are presented in Section 2. The robust 2D DFT forms are developed in Section 3. Numerical example is given in Section 4.

IEEE Int. Conference on Electronics, Circuits and Systems, ICECS 2002, Dubrovnik, Sept.2002.

II. THEORETICAL BACKGROUND

A. Signal filtering

Consider signal $f(n)$ corrupted by a white noise $\nu(n)$, $x(n) = f(n) + \nu(n)$. Filtering can be done by using several samples around the considered one. The filter output can be defined as a value $\hat{f}(n)$ that minimizes the following functional:

$$J(m; n) = \sum_{k=n-M}^{n+M} F(x(k) - m), \quad (1)$$

$$\hat{f}(n) = \arg \min_m J(m; n), \quad (2)$$

where $F(e)$ is the loss function. The maximum likelihood (ML) estimation, for the case of noise with probability density function $p_\nu(\xi)$, is produced by using the loss function $F(e) \approx -\ln p_\nu(e)$. Thus, the ML estimation for the Gaussian noise is obtained by using $F(e) = |e|^2$, resulting in the moving average filter:

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2M+1} \sum_{k=n-M}^{n+M} x(k) \\ &= \text{mean}\{x(k), k \in [n-M, n+M]\}. \end{aligned} \quad (3)$$

The ML estimate for the Laplacian noise is produced by using the median filter. It follows from (1), (2) with the loss function $F(e) = |e|$:

$$\hat{f}(n) = \text{median}\{x(k), k \in [n-M, n+M]\}. \quad (4)$$

The median filter exhibits slightly worse results in the Gaussian noise environment than the moving average filter. However, the moving average filter behavior is poor in the case of impulse noise, while the median filter is very robust to the influence of impulse noise.

B. Robust DFT

Consider the standard DFT:

$$X_S(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}. \quad (5)$$

It can be understood as an average of the modulated samples $\{x(n)e^{-j2\pi nk/N}, n \in [0, N)\}$:

$$X_S(k) = \text{mean}\{x(n)e^{-j2\pi nk/N}, n \in [0, N)\}. \quad (6)$$

The DFT can be obtained as a value that minimizes the functional:

$$J(m; k) = \sum_{n=0}^{N-1} F(x(n)e^{-j2\pi nk/N} - m), \quad (7)$$

$$X_S(k) = \arg \min_m J(m; k), \quad (8)$$

for the loss function $F(e) = |e|^2$. The standard DFT, as its filtering counterpart, the moving average filter, exhibits very poor results in the impulse noise environment. The robust DFT is introduced to overcome the mentioned drawback of the standard DFT. It is defined by Katkovnik as a solution of the minimization problem (7), (8) for the loss function $F(e) = |e|$. This form, known as the robust M -DFT, can be written as [3], [4]:

$$X_R(k) = g(k) \sum_{n=0}^{N-1} \frac{x(n)e^{-j2\pi nk/N}}{|x(n)e^{-j2\pi nk/N} - X_R(k)|}, \quad (9)$$

where

$$g(k) = \left(\sum_{n=0}^{N-1} \frac{1}{|x(n)e^{-j2\pi nk/N} - X_R(k)|} \right)^{-1}. \quad (10)$$

Expressions (9) and (10) are an implicit solution for the robust M -DFT. The iterative procedure for determination of $X_R(k)$ is used in [3], [4].

Separate minimization of the real and imaginary parts of error function $(x(n)e^{-j2\pi nk/N} - m)$ gives the solution known as the marginal median filter [6], [13]:

$$\begin{aligned} X_M(k) = & \text{median}\{\text{Re}\{x(n)e^{-j2\pi nk/N}\}, n \in [0, N)\} + \\ & j \text{median}\{\text{Im}\{x(n)e^{-j2\pi nk/N}\}, n \in [0, N)\}. \end{aligned} \quad (11)$$

The accuracy of the presented robust DFT forms is of the same order of magnitude [6].

III. ROBUST 2D DFT

The standard 2D DFT is defined by:

$$X_S(k_1, k_2) = \frac{1}{N_1 N_2} \times$$

$$\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j2\pi n_1 k_1 / N_1 - j2\pi n_2 k_2 / N_2}, \quad (12)$$

or alternatively as:

$$X_S(k_1, k_2) = \frac{1}{N_1} \sum_{n_1=0}^{N_1-1} X'_S(n_1, k_2) e^{-j2\pi n_1 k_1 / N_1}, \quad (13)$$

where $X'_S(n_1, k_2)$ is the DFT of rows of the matrix $x(n_1, n_2)$:

$$X'_S(n_1, k_2) = \frac{1}{N_2} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-j2\pi n_2 k_2 / N_2}. \quad (14)$$

Expressions (12)-(14) can be written in the following forms:

$$\begin{aligned} X_S(k_1, k_2) = & \text{mean}\{x(n_1, n_2) e^{-j2\pi n_1 k_1 / N_1 - j2\pi n_2 k_2 / N_2}, \\ & n_1 \in [0, N_1), n_2 \in [0, N_2)\}, \end{aligned} \quad (15)$$

$$\begin{aligned} X_S(k_1, k_2) = & \text{mean}\{X'_S(n_1, k_2) e^{-j2\pi n_1 k_1 / N_1}, n_1 \in [0, N_1)\}, \\ & n_2 \in [0, N_2)\}, \end{aligned} \quad (16)$$

where

$$\begin{aligned} X'_S(n_1, k_2) = & \text{mean}\{x(n_1, n_2) e^{-j2\pi n_2 k_2 / N_2}, \\ & n_2 \in [0, N_2)\}, \end{aligned} \quad (17)$$

Relations (15)-(17) are the solutions of the following minimization problems:

$$\begin{aligned} J_1(m; k_1, k_2) = & \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \\ & F_1(x(n_1, n_2) e^{-j2\pi n_1 k_1 / N_1 - j2\pi n_2 k_2 / N_2} - m), \\ X_S(k_1, k_2) = & \arg \min_m J_1(m; k_1, k_2), \end{aligned} \quad (18)$$

$$J_2(m; k_1, k_2) =$$

$$\sum_{n_1=0}^{N_1-1} F_2(X'_S(n_1, k_2)e^{-j2\pi n_1 k_1/N_1} - m),$$

$$X_S(k_1, k_2) = \arg \min_m J_2(m; k_1, k_2), \quad (19)$$

$$J_3(m; n_1, k_2) =$$

$$\sum_{n_2=0}^{N_2-1} F_3(x(n_1, n_2)e^{-j2\pi n_2 k_2/N_2} - m),$$

$$X'_S(n_1, k_2) = \arg \min_m J_3(m; n_1, k_2), \quad (20)$$

for the loss functions $F_1(e) = F_2(e) = F_3(e) = |e|^2$. Note, that minimization problems (18), (19), and (20) result with the same solution, the standard DFT, in this case.

By analogy with the case of 1D signals, the robust 2D DFT can be defined for the loss function $F(e) = |e|$. It can be realized in numerous ways. In the case of the minimization problem (18), it can be defined alternatively by using:

a) Implicit solution and iterative procedure:

$$X_{R_1}(k_1, k_2) = g(k_1, k_2) \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \frac{x(n_1, n_2)}{|d(n_1, n_2; k_1, k_2)|} e^{-j2\pi n_1 k_1/N_1 - j2\pi n_2 k_2/N_2}, \quad (21)$$

where:

$$g(k_1, k_2) = \left(\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \frac{1}{|d(n_1, n_2; k_1, k_2)|} \right)^{-1}, \quad (22)$$

and

$$d(n_1, n_2; k_1, k_2) =$$

$$x(n_1, n_2)e^{-j2\pi n_1 k_1/N_1 - j2\pi n_2 k_2/N_2} - X_{R_1}(k_1, k_2). \quad (23)$$

b) Marginal median approach:

$$X_{M_1}(k_1, k_2) = \text{median}\{\text{Re}\{x(n_1, n_2) \times$$

$$e^{-j2\pi n_1 k_1/N_1 - j2\pi n_2 k_2/N_2}\},$$

$$n_1 \in [0, N_1), n_2 \in [0, N_2)\} +$$

$$j \text{median}\{\text{Im}\{x(n_1, n_2) \times$$

$$e^{-j2\pi n_1 k_1/N_1 - j2\pi n_2 k_2/N_2}\},$$

$$n_1 \in [0, N_1), n_2 \in [0, N_2)\}. \quad (24)$$

Consecutive application of the minimization problems (19) and (20) results in the following solutions:

c) Consecutive iterative procedures:

$$X_{R_2}(k_1, k_2) = g'(k_1, k_2) \times$$

$$\sum_{n_1=0}^{N_1-1} \frac{X'_R(n_1, k_2)e^{-j2\pi n_1 k_1/N_1}}{|X'_R(n_1, k_2)e^{-j2\pi n_1 k_1/N_1} - X_{R_2}(k_1, k_2)|}, \quad (25)$$

where

$$X'_R(n_1, k_2) = g''(k_1, k_2) \times$$

$$\sum_{n_2=0}^{N_2-1} \frac{x(n_1, n_2)e^{-j2\pi n_2 k_2/N_2}}{|x(n_1, n_2)e^{-j2\pi n_2 k_2/N_2} - X'_R(n_1, k_2)|}. \quad (26)$$

d) Consecutive marginal median calculation:

$$X_{M_2}(k_1, k_2) =$$

$$\text{median}\{\text{Re}\{X'_M(n_1, k_2)e^{-j2\pi n_2 k_2/N_2}\},$$

$$n_2 \in [0, N_2)\} +$$

$$j \text{median}\{\text{Im}\{X'_M(n_1, k_2)e^{-j2\pi n_2 k_2/N_2}\},$$

$$n_2 \in [0, N_2)\}, \quad (27)$$

where:

$$X'_M(n_1, k_2) =$$

$$\text{median}\{\text{Re}\{x(n_1, n_2)e^{-j2\pi n_1 k_1/N_1}\},$$

$$n_1 \in [0, N_1)\} +$$

$$j \text{median}\{\text{Im}\{x(n_1, n_2)e^{-j2\pi n_1 k_1/N_1}\},$$

$$n_1 \in [0, N_1)\}. \quad (28)$$

Functions $g'(k_1, k_2)$ and $g''(k_1, k_2)$ in expressions (25) and (26) are given as:

$$g'(k_1, k_2) =$$

$$\left(\sum_{n_1=0}^{N_1-1} \frac{1}{|X'_R(n_1, k_2)e^{-j2\pi n_1 k_1/N_1} - X_{R_2}(k_1, k_2)|} \right)^{-1} \quad (29)$$

and

$$g''(k_1, k_2) =$$

$$\left(\sum_{n_2=0}^{N_2-1} \frac{1}{|x(n_1, n_2)e^{-j2\pi n_2 k_2/N_2} - X'_R(n_1, k_2)|} \right)^{-1}. \quad (30)$$

The solutions $X_{R_2}(k_1, k_2)$ can be obtained through two consecutive iterative procedures, the same as in the case of 1D signal. Form of the solution presented by (27) and (28) corresponds to the separable median filter form. It is used in digital image processing [14].

It can be easily concluded that the presented solutions for $X_{R_i}(k_1, k_2)$, $i = 1, 2$, and $X_{M_i}(k_1, k_2)$, $i = 1, 2$, are not equal to each other. This is due to the nonlinearity of the proposed transforms.

A. Calculation complexity

The median filter based form is more computationally efficient than the iterative procedure in the 1D signal case [6]. The same conclusion holds for the 2D signals. The calculation of the median filter form of the 2D DFT (24) needs $N_1 N_2 \log_2 N_1 N_2$ comparisons for each point or $N_1^2 N_2^2 \times \log_2 N_1 N_2$ comparisons for the entire frequency plane. The separable median forms (27) and (28) need $N_1 \log_2 N_1 + N_2 \log_2 N_2$ comparisons for each point, or $N_1^2 N_2 \log_2 N_1 + N_1 N_2^2 \log_2 N_2$ for the entire frequency plane. This means that the separable median form of the 2D DFT needs an order of magnitude less comparisons than the median filter form. Note, that the standard median behaves better in the image processing application than the separable median [14]. However, the calculation of the DFT needs more samples than the calculation of the median filter in digital image processing. As number of samples increases, the performance of both median forms, (24) and (27), will get closer to each other.

Note, that after calculation of the robust 1D DFT by using the signal samples along one coordinate, by using iterative procedure (26), the second step can be performed by using the marginal median filter form (27), and vice versa. Furthermore, different loss functions $F_2(e)$ and $F_3(e)$ can be applied for minimization of (19) and (20). This can be used to further decrease the calculation complexity. Since the number of samples in the matrix $x(n_1, n_2)$ can be very large, for example $N_1 \times N_2 = 256 \times 256$, application of the robust 1D DFT along one direction will remove all impulses appearing in the matrix. Then the 1D standard DFT can be applied to the other

coordinate. Calculation complexity for this algorithm is reduced to $N_1^2 N_2 \log_2 N_1$ comparisons. Note that calculation complexity of the FFT algorithm exhibits $N_1 N_2 \log_2 N_2$ additions and multiplications, and it can be neglected with respect to the number of comparisons.

IV. NUMERICAL EXAMPLE

Consider the signal:

$$x(t_1, t_2) =$$

$$\begin{aligned} & \exp(j12\pi t_1 + j48\pi t_2) + \exp(j96\pi t_1 - j48\pi t_2) + \\ & \exp(-j48\pi t_1 + j48\pi t_2) + \exp(-j36\pi t_1 - j48\pi t_2), \end{aligned} \quad (31)$$

embedded in a high amount of impulse noise:

$$y(t_1, t_2) = x(t_1, t_2) + a(\nu_1^3(t_1, t_2) + j\nu_2^3(t_1, t_2)), \quad (32)$$

where $a = 10$ and $\nu_i(t_1, t_2)$, $i = 1, 2$ are mutually independent white Gaussian noises with unitary variance $E\{\nu_i(t_1, t_2)\nu_j(t_1, t_2)\} = \delta(i - j)$. The signal is sampled with $\Delta t_1 = \Delta t_2 = 1/128$, along both coordinates. The sampled version of the signal contains 256×256 samples. The standard 2D DFT fails to produce any reasonable result in this case (Figure 1(a)). However, realization of the standard 2D DFT is very fast and needs only 0.1sec on a PC Pentium IV. The robust 2D DFT based on the median filter (24) is shown in Figure 1(b). Necessary time for its calculation was 2h 47min 4.6sec. However, large portion of calculation time has been taken by reshaping matrix into the vector suitable for the median determination. The separable median realization (27), (28) is shown in Figure 1(c). The calculation time was 27.6sec. The robust 2D DFT, obtained by using the robust DFT along one coordinate and the standard DFT along the other, is shown in Figure 1(d). Its realization has taken 18.1sec. It can be seen that all forms of the robust 2D DFT produce similar accuracy in this example, while the last one is calculated in shortest time period.

V. CONCLUSION

Generalization of the robust DFT form to the 2D signals is presented. various forms

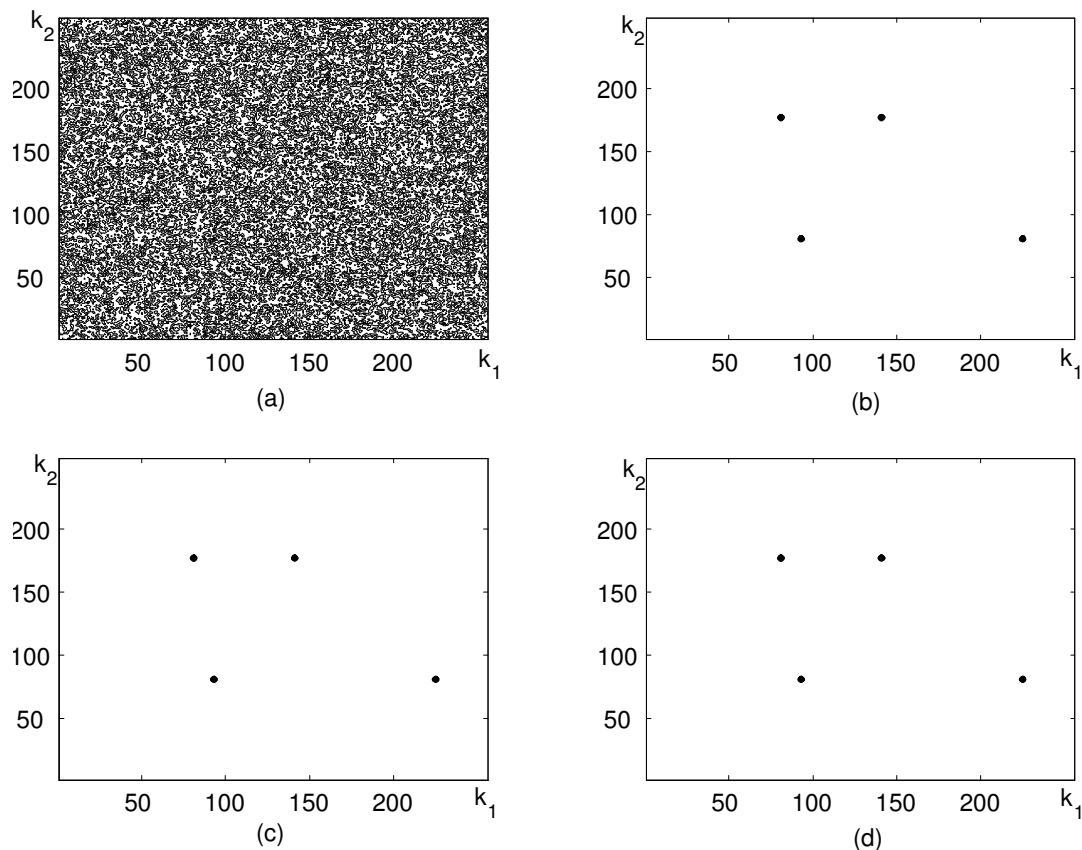


Fig. 1. 2D DFT forms of signal embedded in impulse noise: (a) Standard 2D DFT; (b) Robust 2D DFT - marginal median filter form; (c) Robust 2D DFT - separable median filter form; (d) Robust 2D DFT - robust 1D DFT along one coordinate and the standard 1D DFT along the other.

of the robust 2D DFT are introduced. The fastest one assumes the calculation of the robust DFT along the rows or columns of the matrix, followed by the application of the standard DFT along the other direction. The straightforward generalization to the case of multidimensional signals is possible.

VI. ACKNOWLEDGMENT

The work of I. Djurović is supported by the Postdoctoral Fellowship for Foreign Researchers of Japan Society for the Promotion of Science and the Ministry of Education, Culture, Sports, Science and Technology under Grant 01215. The work of LJ. Stanković and J.F. Böhme is supported by the Volkswagen

Stiftung, Federal Republic of Germany.

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