Analysis of Noise in Time-Frequency Distributions

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***Abstract*—Exact expressions for the quadratic distributions' variance of signals corrupted with white stationary, white nonstationary, and colored stationary noise are derived. It has been shown that the signal-dependent part of variance is closely related to the nonnoisy distribution values.**

***Index Terms*—Eigenvalues and eigenfunctions, noise, signal representations, spectral analysis, time-frequency analysis, Wigner distribution.**

I. Introduction

INFLUENCE OF NOISE on time-frequency distributions is studied in [1]–[3]. Spectral estimators of time-varying processes have been considered in [4]–[6]. The mean variance values for noisy signals are derived in [1], [3]. An algorithm for optimization of parameters in the Wigner distribution (WD) and in other quadratic distributions of noisy signals has been proposed [8]. In this letter, the exact expressions for the variance of quadratic distributions in the cases of stationary and nonstationary white noise and colored stationary noise are derived. It has been shown that the signal-dependent part of variance is equal to a new quadratic distribution of nonnoisy signals.

II. Noise in Quadratic Time-Frequency Distributions

Discrete-time form of the Cohen class of distributions (CD) of noise $\nu(n)$ is defined by

$$C_\nu(n,\omega;\varphi)=\sum_k\sum_m \varphi(m,k)\nu(n+m+k)\cdot\nu^*(n+m-k)e^{-j2\omega k} \quad (1)$$

where $\varphi(m,k)$ is the kernel in time-lag domain [1], [7], [9]. All summation limits are from $-\infty$ to ∞, unless indicated otherwise.

Variance of the CD, for a complex Gaussian noise with independent identically distributed real and imaginary part (i.i.d.), is [1], [3]

$$\begin{aligned}\sigma^2_{\nu\nu}(n,\omega)=&\sum_{k_1}\sum_{k_2}\sum_{m_1}\sum_{m_2}\varphi(m_1,k_1)\,\varphi^*(m_2,k_2)\\ &\cdot R_{\nu\nu}(n+m_1+k_1,n+m_2+k_2)\\ &\cdot R^*_{\nu\nu}(n+m_1-k_1,n+m_2-k_2)\\ &\cdot e^{-j2\omega(k_1-k_2)}\end{aligned} \quad (2)$$

where $R_{\nu\nu}(m,n)$ is the noise autocorrelation function and $R_{\nu^*\nu}(n,m)\equiv 0$.

For a *nonstationary complex white noise*, whose autocorrelation function is $R_{\nu\nu}(m,n)=I(n)\delta(m-n)$, $I(n)\geq 0$, we get

$$\begin{aligned}\sigma^2_{\nu\nu}(n,\omega)&=\sum_k\sum_m|\varphi(m,k)|^2I(n+m+k)I^*(n+m-k)\\ &=C_I\left(n,0;|\varphi|^2\right).\end{aligned} \quad (3)$$

The stationary case follows with $I(n)=\sigma^2_\nu$.

If the noise $\nu(n)$ is *complex colored stationary* $R_{\nu\nu}(m,n)=R_{\nu\nu}(m-n)$, then its Fourier transform $N(\theta)$ is a white nonstationary noise, with the autocorrelation function in frequency domain $R_{NN}(\theta_1,\theta_2)=2\pi S_{\nu\nu}(\theta_1)\delta(\theta_1-\theta_2)$. (From $R_{NN}(\theta_1,\theta_2)=\sum_m\sum_n E\{\nu(m)\nu^*(n)\}\exp(-j\theta_1 m+j\theta_2 n)$, with $E\{\nu(m)\nu^*(n)\}=R_{\nu\nu}(m-n)$, we get $R_{NN}(\theta_1,\theta_2)=\sum_k\sum_n R_{\nu\nu}(k)\exp(-j\theta_1 k+j(\theta_2-\theta_1)n)=2\pi S_{\nu\nu}(\theta_1)\delta(\theta_1-\theta_2)$.) Thus, a form that is dual to (3) holds in this case

$$\begin{aligned}\sigma^2_{\nu\nu}(n,\omega)=&\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|P(\theta,\xi)|^2\,S^*_{\nu\nu}\left(\omega-\frac{\xi}{2}+\frac{\theta}{2}\right)\\ &\cdot S_{\nu\nu}\left(\omega-\frac{\xi}{2}-\frac{\theta}{2}\right)d\theta d\xi\\ =&C_{S_{\nu\nu}}\left(0,\omega;|P|^2\right)\end{aligned} \quad (4)$$

where $P(\theta,\xi)$ is the kernel in Doppler-frequency domain. Relation (4) can also be obtained by using the fact that (2), for stationary noise, can be written as

$$\begin{aligned}\sigma^2_{\nu\nu}(n,\omega)=&\sum_k\sum_m\varphi(m,k)\\ &\cdot\left\{\varphi(m,k)^{**}_{m,k}\left[R^*_{\nu\nu}(m+k)R_{\nu\nu}(m-k)e^{j2\omega k}\right]\right\}^*\end{aligned}$$

where $^{**}_{m,k}$ denotes a two-dimensional (2-D) convolution. By applying 2-D Parseval's theorem we get (4). It is important to note that the transforms in (4) are periodic in ξ and θ, with period 2π [3], [12]. Special cases, e.g., analytic noise, easily follow from (4) (see [3]).

III. Noisy Signals

For deterministic signals $f(n)$ corrupted by noise $x(n)=f(n)+\nu(n)$, the distribution variance consists of two parts [1], [3]:

$$\sigma^2_{CD}(n,\omega)=\sigma^2_{\nu\nu}(n,\omega)+\sigma^2_{f\nu}(n,\omega). \quad (5)$$

The first part has already been studied in Section II and in [1] and [3], and the mean value of the second part is derived in [1]

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and [3]. In order to get the exact value of $\sigma^2_{f\nu}(n,\omega)$, we will use the inner product form of the CD [9], [10]:

$$C_x(n,\omega;\psi) = \sum_m \sum_k \psi(m,k) \cdot \left[x(n+m)e^{-j\omega m}\right]\left[x(n+k)e^{-j\omega k}\right]^* \quad (6)$$

where $\psi(m,k) = \varphi((m+k)/2,(m-k)/2)$. Calculation of $\psi(m,k)$ is described within the numerical example. For a real and symmetric $\varphi(m,k)$ and complex (i.i.d.) noise we get

$$\begin{aligned}\sigma^2_{f\nu}(n,\omega) =& 2\sum_{k_1}\sum_{k_2}\sum_{m_1}\sum_{m_2}\psi(m_1,k_1)\psi^*(m_2,k_2) \\ &\cdot f(n+m_1)f^*(n+m_2) \\ &\cdot R_{\nu\nu}(n+k_2,n+k_1) \\ &\cdot e^{j\omega(m_2-k_2-m_1+k_1)} \\ =& 2\sum_{m_1}\sum_{m_2}\phi(m_1,m_2)\left[f(n+m_1)e^{-j\omega m_1}\right] \\ &\cdot\left[f(n+m_2)e^{-j\omega m_2}\right]^* \\ =& 2C_f(n,\omega;\phi). \end{aligned} \quad (7)$$

General form of the new kernel $\phi(m_1,m_2)$ is

$$\phi(m_1,m_2) = \sum_{k_1}\sum_{k_2}\psi(m_1,k_1)\psi^*(m_2,k_2) \cdot e^{-j\omega(k_2-k_1)}R_{\nu\nu}(n+k_2,n+k_1). \quad (8)$$

Note: The signal-dependent part of variance $\sigma^2_{f\nu}(n,\omega)$ is equal to a new quadratic distribution of the signal $f(n)$ with the kernel $\phi(m_1,m_2)$.

A. Special Cases

1) White Stationary Complex Noise: White stationary complex noise $R_{\nu\nu}(n+k_1,n+k_2) = \sigma^2_\nu\delta(k_1-k_2)$ produces

$$\phi(m_1,m_2) = \sigma^2_\nu\sum_k\psi(m_1,k)\psi^*(m_2,k). \quad (9)$$

For finite limits, this is a matrix multiplication form, $\mathbf{\Phi} = \sigma^2_\nu\mathbf{\Psi}\mathbf{\Psi}^* = \sigma^2_\nu\mathbf{\Psi}^2$, since for time-frequency kernels $\psi^*(m_2,k) = \psi(k,m_2)$ holds. The boldface letters, without arguments, are used to denote matrices. For example, $\mathbf{\Psi}$ is a matrix with elements $\psi(m,k)$. Thus,

$$\sigma^2_{f\nu}(n,\omega) = 2\sigma^2_\nu C_f\left(n,\omega;\mathbf{\Psi}^2\right). \quad (10)$$

Some interesting conclusions that can be drawn from (9) and (10) are presented in the Appendix.

a) Eigenvalue decomposition: Assume that both the summation limits and the values of $\psi(m,k)$ are finite. It is true when the kernel $\varphi(m,k)$ is calculated from the well-defined kernel in a finite Doppler-lag domain $\varphi(m,k) = FT_\theta\{c(\theta,k)\}$ using a finite number of samples. The signal-dependent part of variance $\sigma^2_{f\nu}(n,\omega)$ can be calculated, like other distributions from the Cohen class, by using eigenvalue decomposition of $\mathbf{\Psi}$ [9], [10]. The distribution (6) of nonnoisy signal $f(n)$ is

$$C_f(n,\omega) = \sum_{i=-N/2}^{(N/2)-1}\lambda_i S_f(n,\omega;q_i) = C_f(n,\omega;\lambda,q) \quad (11)$$

where λ_i and $q_i(m)$ are the eigenvalues and eigenvectors of $\mathbf{\Psi}$, and

$$S_f(n,\omega;q_i) = \left|\sum_{m=-N/2}^{(N/2)-1}f(n+m)q_i(m)e^{-j\omega m}\right|^2$$

is the spectrogram of $f(n)$, with $q_i(m)$ playing the role of window. Since $\mathbf{\Phi} = \sigma^2_\nu\mathbf{\Psi}^2$, its eigenvalues and eigenvectors are $\mu_i = \sigma^2_\nu|\lambda_i|^2$ and $q_i(m)$. Thus, we have

$$\begin{aligned}\sigma^2_{f\nu}(n,\omega) &= 2\sigma^2_\nu\sum_{i=-N/2}^{(N/2)-1}|\lambda_i|^2 S_f(n,\omega;q_i) \\ &= 2\sigma^2_\nu C_f\left(n,\omega;|\lambda|^2,q\right).\end{aligned} \quad (12)$$

b) Relation between the original kernel and variance $\sigma^2_{f\nu}(n,\omega)$ kernel: According to (11), we can conclude that the original kernel in Doppler-lag domain can be decomposed as $c(\theta,k) = \sum_{i=-N/2}^{(N/2)-1}\lambda_i a_i(\theta,k)$, where $a_i(\theta,k)$ are orthonormal two-dimensional basis functions (ambiguity functions of the eigenvectors $q_i(m)$). The kernel of $C_f(n,\omega;|\lambda|^2,q)$ in (12) is $c_\sigma(\theta,k) = \sum_{i=-N/2}^{(N/2)-1}|\lambda_i|^2 a_i(\theta,k)$. A detailed analysis of distributions, with respect to their eigenvalue properties, is presented in [9]. In that sense, the signal-dependent variance is just "an energetic map of the time-frequency distribution" of the original signal [9].

2) Nonstationary White Noise: For nonstationary white noise, we have

$$\phi(m_1,m_2) = \sum_k I(n+k)\psi(m_1,k)\psi^*(m_2,k) \quad (13)$$

i.e., $\mathbf{\Phi} = \mathbf{\Psi}\mathbf{I}_n\mathbf{\Psi}^*$, where $\mathbf{I}_n$ is a diagonal matrix, with the elements $I(n+k)$. For the quasi-stationary case, $I(n+k_1)\delta(k_1-k_2) \cong I(n)\delta(k_1-k_2)$, we have $\mu_i = I(n)|\lambda_i|^2$, with all other parameters as in (12).

3) Colored Stationary Noise: In the case of colored stationary noise, a relation dual to (13) holds [see (3) and (4)].

4) Numerical Example: Consider the signal

$$\begin{aligned}x(t) =& \exp\left(j\left(-60\sin(3\pi t)+400t^2+1500t\right)\right) \\ &+e^{-25(t-0.67)^2}\exp\left(j1200(t-0.4)^2\right)+\nu(t)\end{aligned} \quad (14)$$

within the interval [0, 1], sampled at $\Delta t = 1/1024$. The Hanning window of the width $T_w = 1/4$ is used. A high stationary complex white noise, with variance $\sigma^2_\nu = 2$, is assumed. The WD, spectrogram, S-method [3], [4], and Choi–Williams distribution (CWD) are presented in Fig. 1(a)–(d), respectively. Sampling in the WD is $\Delta t/2$. The CWD kernel

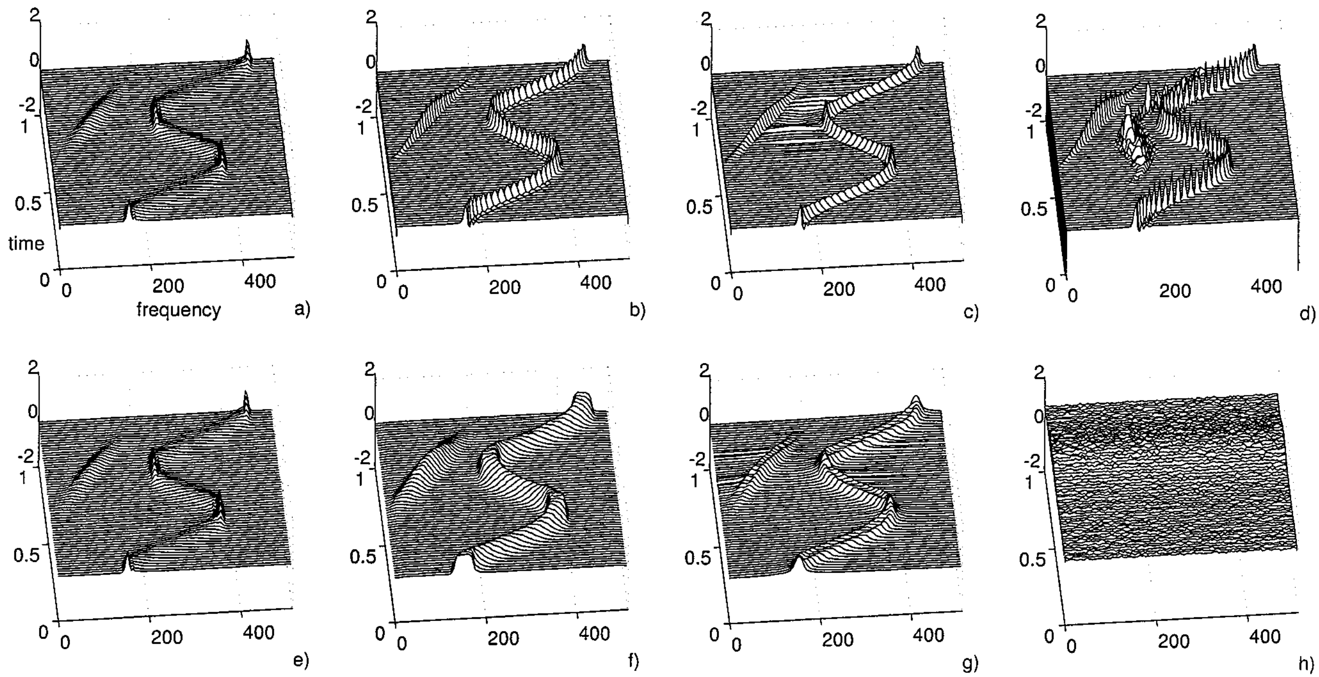

Fig. 1. Time-frequency representations of a nonnoisy signal and variances of the noisy signal. (a) Spectrogram. (b) S-method with $L = 5$. (c) Choi–Williams distribution. (d) Pseudo-Wigner distribution. (e) Variance in the spectrogram. (f) Variance in the S-method. (g) Variance in the Choi–Williams distribution. (h) Variance in the pseudo-Wigner distribution. Variances are statistically obtained by averaging over 2000 realizations. They almost coincide with the values obtained by evaluating the derived expressions (3) and (12). Normalized values are presented.

$c(\theta,\tau) = \exp(-(\theta\tau/2\pi)^2)$ within $-\sqrt{\pi N/2} \leq |\theta|, |\tau| \leq \sqrt{\pi N/2}$, and 128 samples along each axis is used. Elements of matrix $\mathbf{\Psi}$ are calculated based on $c(\theta,\tau)$ as

$$\psi(m,n) = \sum_{p=-N/2}^{(N/2)-1} c(p\Delta\theta,(m-n)\Delta\tau)e^{-j2\pi(m+n)p/(2N)}\Delta\theta.$$

Normalized eigenvalues of the CWD kernel are $\lambda_i = \{1, -0.87, 0.69, -0.58, 0.41, -0.30, \ldots\}$ and $\mu_i = |\lambda_i|^2 = \{1, 0.76, 0.47, 0.33, 0.17, 0.09, \ldots\}$. In the spectrogram, the whole signal-dependent part of variance is "located" just on the signal components, and it is proportional to the spectrogram values [Fig. 1(e)]. Variance in the S-method is proportional to the sum of several frequency-shifted spectrograms [Fig. 1(f)]. Obviously, the variance $\sigma^2_{f\nu}(n,\omega)$ in the CWD is a sum of few spectrograms with positive weighting coefficients μ_i and windows $q_i(m)$ [Fig. 1(g)]. In the WD, the variance is "spread" over the entire time-frequency plane [Fig. 1(h)].

IV. Conclusion

We study the influence of additive noise on time-frequency analysis. The variance has two parts. The signal-dependent one is equal to a new quadratic time-frequency distribution, whose kernel can be calculated based on the original distribution kernel. For the spectrogram, this part of variance is proportional to the spectrogram of the original signal without noise. In the Wigner distribution, both the signal-dependent and noise-only-dependent parts of the variance are uniformly spread over the entire time-frequency plane. For reduced interference distributions, the signal-dependent part of variance is distributed at and around the autoterms, meaning that it can dominantly influence time-frequency-based analysis, even for very low input signal-to-noise ratio.

Appendix

Any two distributions whose kernels satisfy relation $\psi_1(m,k) = \psi_2(m,-k)$ and $\psi_1(m,k) = \psi_1(k,m)$ have the same variance (10), since

$$\sum_k \psi_1(m_1,k)\psi_1^*(m_2,k) = \sum_k \psi_1(m_1,-k)\psi_1^*(m_2,-k) = \sum_k \psi_2(m_1,k)\psi_2^*(m_2,k).$$

A. Corollary

A distribution with product kernel $c(\theta\tau)$ and the distribution with its dual kernel $c_d(\theta\tau) = FT_{2D}\{c(\alpha\beta)\}$ have the same variance.

B. Example

1) The WD has the kernel $c(\theta\tau) = 1$, i.e., $\psi(m,k) = \delta(m+k)$. According to the corollary, the WD has the same variance as its dual kernel counterpart, with kernel $c(\theta\tau) = \delta(\theta,\tau)$, i.e., $\psi(m,k) = \delta(m-k)$. The last kernel corresponds to the signal energy $\sum_m |x(n+m)|^2$ [see (6)].
2) The pseudo-WD $\psi(m,k) = w(m)\delta(m+k)w(k)$ and the mean value of spectrogram over frequency $\psi(m,k) = w(m)\delta(m-k)w(k)$ have the same variance.

3) The same holds for the smoothed spectrogram and for the S-method, whose kernels are $\psi(m,k) = w(m)p(m+k)w(k)$ and $\psi(m,k) = w(m)p(m-k)w(k)$, respectively [3], [4]. Their variance is $\sigma^2_{f\nu}(n,\omega) = 2\sigma^2_\nu \sum_{i=-L}^{L} \text{SPEC}_f(n,\omega - i\Delta\omega)$ for $P(k) = FT\{p(n)\} = 1$, $i = -L,\ldots,0,\ldots,L$ and 0 elsewhere [Fig. 1(e) and (f)] [11].

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