

# Compressive Sensing Based Separation of Nonstationary and Stationary Signals Overlapping in Time-Frequency

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**Abstract**—A compressive sensing (CS) approach for nonstationary signal separation is proposed. This approach is motivated by challenges in radar signal processing, including separations of micro-Doppler and main body signatures. We consider the case where the signal of interest assumes sparse representation over a given basis. Other signals present in the data overlap with the desired signal in the time and frequency domains, disallowing conventional windowing or filtering operations to be used for desired signal recovery. The proposed approach uses linear time-frequency representations to reveal the data local behavior. Using the L-statistics, only the time-frequency (TF) points that belong to the desired signal are retained, whereas the common points and others pertaining only to the undesired signals are deemed inappropriate and cast as missing samples. These samples amount to reduced frequency observations in the TF domain. The linear relationship between the measurement and sparse domains permits the application of CS techniques to recover the desired signal without significant distortion. We focus on sinusoidal desired signals with sparse frequency-domain representation but show that the analysis can be straightforwardly generalized to nonsinusoidal signals with known structures. Several examples are provided to demonstrate the effectiveness of the proposed approach.

**Index Terms**—Compressed sensing, time-frequency analysis, L-statistics, components separation.

## I. INTRODUCTION

COMPRESSIVE sensing (CS) considers reconstruction of signals that have sparse basis representations in a certain transform domain, using an incomplete set of samples [1]–[11]. In many applications, the signals of interest are sparse in the Fourier-domain. In this case, the application of conditioned  $\ell_1$  norm minimization, using a reduced number of time-domain observations, has produced impressive signal reconstructions. However, other applications, including radar, give rise to nonstationary signals that are characterized by instantaneous fre-

quency laws. These signals are not typically narrowband and, as such, cannot be cast as sparse over the frequency variable.

In this paper, we deal with a class of problems where the missing samples or observations are not due to Nyquist sampling relaxation, but rather occur as a consequence of attempting to separate desired from undesired nonstationary signal components. If these components highly overlap in time and frequency, signal separation and desired signal recovery cannot be accomplished through conventional methods involving windowing or filtering. Furthermore, separation in the TF domain becomes difficult if the respective TF signatures reside over common TF regions or encounter several TF intersection points. This makes TF masking difficult and renders TF synthesis methods ineffective.

A CS approach dealing with nonstationary signals was introduced in [6], [7], with the objective of achieving high resolution quadratic TF distributions. The underlying assumption is that multi-component signals with instantaneous frequency characterizations are considered sparse in the joint TF representation. Instead of using the classical reduced interference kernels with low-pass filter characteristics [12], [13], a few samples of the ambiguity function around the origin are selected to constitute reduced observations [6], and thus forming a "CS ambiguity function". By applying  $\ell_1$  reconstruction algorithms, a sparse TF representation is achieved with high energy concentration and reduced cross-terms. Both the approach in [6] and our proposed approach are based on a common premise, namely, the missing observations are undesired samples, which are removed from consideration due to interference contribution. However, the approach in [6], which is based on bilinear TF representation, does not consider signal separation or recovery. Moreover, the methods based on quadratic distributions, generally fail to recover signal phase. Rather, data observations selection is guided in [6] by reduced cross-terms and high TF localization, neither of which is a leading motivation in our paper. The problem statement, formulation and application are quite different: we deal with signal separation when significant signal components overlap in time-frequency plane. The overlapping regions in time-frequency plane will not produce cross-terms in the ambiguity plane that are distant from the origin. In the ambiguity function these components, overlapping simultaneously in both time and frequency will be considered as a single component. Ambiguity domain analysis, including the CS, will not be able to deal with these signal behaviors. In addition, in the proposed approach the signal phases can be recovered accurately.

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The approach presented in [14] attempted at separating sinusoidal signals from others by only considering the ambiguity function points along the zero time-lag. This approach was introduced in the context of direction finding and nonstationary array signal processing, and does not deal with the CS problem formulation. Further, it only focuses on chirp signals, as the undesired signal components, and is not aimed at signal recovery or reconstruction.

The main goal of this paper is to use the CS approach to recover narrowband signals when contaminated with highly nonstationary signals. In this case, the desired sparse representation is achieved using the Fourier basis. For narrowband signals, the unknown signal parameters may include the signal amplitudes, frequencies, phases, as well as the number of components. However, unlike the standard CS formulations, involving sparse Fourier domain, here the observations are made in the TF domain rather than in the time domain. The samples in the TF domain are selected to favor the sinusoidal signals, which are both locally and globally sparse. For these signals, a dictionary can be defined which relates the global sparse representation in the Fourier-domain to the local spectral sparseness, when viewed through a short time-window. The undesired nonstationary signals, on the other hand, may be spectrally sparse over the same window, but loses their sparseness property when considered over the entire data record. The local behavior of both types of signals is revealed by taking the short time Fourier transform (STFT). The time-frequency regions corresponding to the nonstationary signals over all windows are identified and removed from consideration, and therefore, are cast as missing observations. For a successful removal of these regions, the L-statistics based analysis [22]–[24] are tailored to the time-frequency representation. The L-estimation filters (L-filters) and general Huber estimation theory [23] have attracted much attention in the signal and image processing [3], [4]. In order to be able to deal with impulsive and nonimpulsive noise components, the L-estimators are defined as linear combinations of order statistics [22], [28]. The L-statistics based realizations of signal transforms and representations are presented in [22], [29], [32]. Here, we use the L-statistics to separate overlapping and nonoverlapping time-frequency regions before the CS methods are applied.

The theory presented in this paper is inspired by numerous practical applications. For example, in radar signal processing, micro-Doppler effects can obscure rigid body points [15]–[18], rendering the radar image highly cluttered and unreadable. In this case, the micro-Doppler effects are represented by highly nonstationary signals, while the rigid body points are represented by sparse Fourier-domain signals (weighted sum of fixed frequency sinusoids), [15]. A simplified example is presented in Fig. 1, where the micro-Doppler is induced by four rotating parts reflecting continuously and several rotating flashing parts. The sparse (rigid body) signals intersect with the nonstationary components at various TF points, as evident in Fig. 1. We remove a large number of overlapping points or intervals, and retain only those TF observations belonging to the narrowband signals. Similar situation may, for example, arise in communications, when narrowband signals are disturbed by a frequency hopping jammer that is of shorter duration than the considered

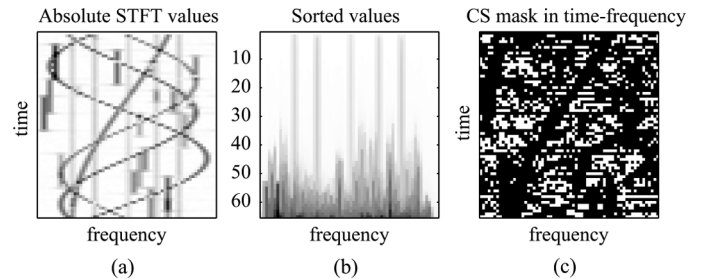


Fig. 1. Time-frequency representation of the radar signal, at a given range, corresponding to five rigid body points, four fast rotating reflectors and several points reflecting during a short time interval, producing flashes: (a) The absolute STFT value within the coherent integration time, (b) The sorted absolute STFT values, along time, for each frequency (cross-range), (c) The matrix showing available (in white) and omitted (in black) values in the time-frequency plane, after the L-statistics approach with 60% of omitted values is used. The Hanning window is used in time-frequency analysis.

time-interval, but may also be overlapping with narrowband signals within same intervals.

The paper is organized as follows. In Section II, the problem formulation in TF plane is provided. A separation technique, based on the L-statistics, is presented in this section. Signal reconstruction, based on the compressive sensed TF matrices, is discussed in Section III. Cases of time and frequency varying windows as well as the cases of overlapping and nonoverlapping STFTs are presented in the form suitable for the CS techniques application. Supporting examples are given in Section IV. Section V is the Conclusion.

## II. FORMULATION OF PROBLEM IN TIME-FREQUENCY DOMAIN

The desired signal is assumed sparse in a given basis (in frequency domain), and as such can be recovered from few observations. In this respect, we consider the desired signal to be a weighted sum of unknown number of sinusoids with unknown parameters. If the undesired and desired signal components highly overlap in time and frequency, signal removal or separation using filtering and windowing techniques becomes ineffective. Accordingly, data observations, drawn from the output of either technique, would include large interference components, preventing accurate reconstruction of desired signal using  $\ell_1$  norm minimizations. However, if the undesired signal is localizable in the TF domain, then one can identify TF points that are free from interference or where the interference is least significant. CS techniques, based on these TF points, can then be efficiently applied for sparse signal recovery. The difficulty of masking out the interference without losing portions of the desired signal impedes the use of typical TF synthesis methods for accurate signal recovery [19]–[21].

The analyzed problem can be stated as follows. Let us consider a composite signal,

$$x(n) = x_{sp}(n) + x_{ns}(n) \quad (1)$$

where  $x_{sp}(n)$  is the stationary and sparse signal part and  $x_{ns}(n)$  is the highly nonstationary part. This kind of signal composition is inspired by the radar signal returning from moving targets,

where both a rigid body and micro-Doppler components can be present. The DFT of this signal can, therefore, be defined as:

$$X(k) = X_{sp}(k) + X_{ns}(k),$$

$$X_{sp}(k) \neq 0 \text{ for } k \in \{k_{01}, k_{02}, \dots, k_{0K}\}, K \ll N \quad (2)$$

where  $N$  is the total number of signal samples. Thus,  $X_{sp}(k)$  is stationary and sparse in  $k$ , whereas  $X_{ns}(k)$  is nonstationary and nonsparse and could assume nonzero values even for all frequencies  $k$ . In the above signal model, the sparse and nonstationary part of signal may significantly overlap in frequency. Further, certain frequency components in  $X_{ns}(k)$  could be much stronger than their counterparts in  $X_{sp}(k)$ ,

$$|X_{ns}(k_{0i})| \gg |X_{sp}(k_{0i})|, i = 1, 2, \dots, K. \quad (3)$$

The nonstationary components at some frequencies  $k_j \neq k_{0i}, i = 1, 2, \dots, K$  could also be much stronger than the signal components at the desired frequencies,  $|X_{ns}(k_j)| \gg |X(k_{0i})|$ .

Classical spectrum analysis tools, dealing with signals separation would prove ineffective in handling signals adhering to the model in (2), even if we assume that the frequencies of desired signals  $k_{01}, k_{02}, \dots, k_{0K}$  are known. For instance, consider an ideal notch filter (its inverse form) applied to a priori known desired frequency (which is not the case in practice). When disturbance appears at the same frequency as the desired component, it cannot be filtered out, even with the use of an ideal notch filter. In this case, the signal amplitude obtained at the filter output will have an incorrect value, as will be illustrated in the examples.

On the other hand we also assume that the sparse part of the signal  $x_{sp}(n)$  and the highly nonstationary part  $x_{ns}(n)$  overlap in a significant number of time samples, i.e., that for any (or all)  $n$ , may hold  $x(n) \neq x_{sp}(n)$ . This renders time-domain signal separation difficult, if not impossible. It prevents the application of the CS algorithms in the time domain. Finally, since we actually have the full set of data available, we may consider the application of  $l_1$  norm based robust processing, which will again fail to extract the sparse part due to time and frequency overlapping with sporadically stronger nonstationary part.

The highly nonstationary signal components invite TF signal representation, which is given by:

$$\rho_x(n, k) = \rho_{sp}(n, k) + \rho_{ns}(n, k), \quad (4)$$

where  $\rho_x(n, k)$  is a linear TF representation (quadratic representations give rise to cross-terms). We deal with the general case where  $\rho_{sp}(n, k)$  and  $\rho_{ns}(n, k)$  overlap in a significant number of points  $(n, k)$  in the TF plane. The significant number of disturbance values in the overlapping regions can be much stronger than their sparse signal counterparts. The reconstruction of desired sparse signal is considered impossible unless we discard the overlapping values. Thus, we must disregard a significant number of TF regions, being left with a much smaller number of independent data  $N_Q$  than the original number of input data  $N$  ( $N_Q \ll N$ ). Consequently, a simple TF based time-varying filtering cannot provide an efficient solution. An effective way to proceed is to apply the CS methods tailored to the time-frequency problem formulation. Note that satisfactory results will

be obtained as long as the number of available data is significantly larger than sparsity of signal ( $N_Q \gg K$ ).

In the TF analysis, we start with the simplest representation based on the STFT in order to assure some important properties: (a) A linear relationship between the sparse and observation domains, (b) Signal localization in the TF domain, preserving the phase information; (c) No presence of cross-terms which arise due to the bilinear products in quadratic TF distributions. In the case of nonoverlapping windows STFTs, the number of available time-frequency samples stays the same as that of input signal ( $N$  samples). Later, the approach will be extended to the overlapping window STFTs. In this case, we still deal with  $N$  signal samples. In other words, although the problem may appear as over-determined, a higher number of overlapped STFTs (compared to nonoverlapped ones) does not mean more available samples, but rather more available linear combinations of the same  $N$  input data samples. This may only affect computation complexity, with some benefits from convergence of rounded windows used in overlapped STFTs.

It is evident from the above problem description that the main challenge lies in properly selecting the signal observations in the STFT domain, avoiding all disturbances, and in establishing a linear relationship between TF and Fourier domain. A technique that could be adapted for this kind of problems is based on the L-statistics. The L-statistics has been already used for realizations of signal transforms and representations when signal is heavily corrupted by impulse noise. Here, the L-statistics is adapted to properly select time-frequency regions. It involves sorting out data samples along time axis for a given frequency and then removing some of them. If we have only the nonstationary component at specific frequency in TF representation, it is clear that eliminating the highest values along that frequency would actually eliminate the undesired nonstationary interference. For the frequency lines with contributions from both the nonstationary interference and the desired sparse sinusoidal components, the highest values would correspond to common or overlapping regions. Removing most of the highest values along the frequency line will remove the interference contribution as well. Another possible case is when the nonstationary and sparse signals are of the same order of amplitude, but the opposite phases produce low values at the intersection points. In this case, the solution is to remove some of the lowest values, in addition to the highest ones. As such, we avoid interference contamination by keeping the middle part of sorted values. The L-statistics is applied to  $\rho_x(n, k)$ , separately for each frequency  $k = k_i$ :

$$L\rho_x(n, k_i) = \mathcal{L}\{\rho_x(n, k_i)\} = \Psi_{k_i}(p)$$

$$\text{for } p \in [\alpha N, \beta N], \text{ where}$$

$$\Psi_{k_i}(p) = \text{sort}_n\{\rho_x(n, k_i)\}, n, p \in \{0, \dots, N-1\}, \quad (5)$$

where  $L$  denotes the L-estimation operator, while the parameters  $\alpha$  and  $\beta$  are defined as  $0 \leq \alpha < \beta \leq 1$ . It means that for each  $k$ , instead of the original  $N$  points,  $n = 0, \dots, N-1$ , we are left with a certain arbitrary set of time intervals where transformation values are used in further calculation:

$$N_1(k_i), N_2(k_i), \dots, N_B(k_i).$$

These intervals will be referred as the frequency dependent arbitrary positioned time intervals. The signal values that appear within these intervals can be considered as available CS measurements, containing the stationary and sparse part of the signal only. Namely, after the L-statistics we have:

$$L\rho_{ns}(n, k) \ll L\rho_{sp}(n, k) \quad \text{for } n \text{ within } N_1(k) \text{ or } N_2(k) \text{ or } \dots \text{ or } N_B(k), \quad (6)$$

where the L-statistics outputs  $L\rho_x(n, k)$  belonging to the non-stationary and sparse components are denoted as  $L\rho_{ns}(n, k)$  and  $L\rho_{sp}(n, k)$ , respectively. Note that,  $L\rho_{ns}(n, k)$  is negligible, while  $L\rho_{sp}(n, k)$  represents the available sparse signal TF values. For example, using the L-statistics with  $Q = 70\%$  ( $Q = 100(1 - \beta + \alpha)$ ) omitted values, for a given  $k$ , implies that the total duration of intervals  $N_1(k), N_2(k), \dots, N_B(k)$  where the sparse signal TF values are available, for a given frequency, is just 30% of the original observations.

The proposed theory can be easily generalized for other domains where the signal is sparse. The Fourier transform domain is just one possibility, corresponding to a weighted sum of sinusoids whose number and parameters are unknown. In general, it is sufficient that the behavior of this part of signal is with known structures (and sparse in any other domain). For example, if we can assume that  $x_{sp}(n)$  changes linearly along frequency in the TF plane with a rate  $a$  then we can easily adjust the method to be used with:

$$\rho_x(n, k - an) \rightarrow \rho_x(n, k). \quad (7)$$

The same holds for any other known structures, up to the unknown set of parameters.

### III. RECONSTRUCTION BASED ON THE CS METHODS

Consider a discrete-time signal  $x(n)$  of the length  $N$  and its discrete Fourier transform (DFT)  $X(k)$ . The STFT, with a rectangular window of the width  $M$ , is:

$$STFT(n, k) = \sum_{m=0}^{M-1} x(n+m)e^{-j2\pi mk/M}. \quad (8)$$

In a matrix form, it can be written as:

$$\mathbf{STFT}_M(n) = \mathbf{W}_M \mathbf{x}(n), \quad (9)$$

where  $\mathbf{STFT}_M(n)$  and  $\mathbf{x}(n)$  are vectors:

$$\begin{aligned} \mathbf{STFT}_M(n) &= [STFT(n, 0), \dots, STFT(n, M-1)]^T, \\ \mathbf{x}(n) &= [x(n), x(n+1), \dots, x(n+M-1)]^T, \end{aligned} \quad (10)$$

and  $\mathbf{W}_M$  is the  $M \times M$  DFT matrix with coefficients:

$$W_M(m, k) = \exp(-j2\pi km/M).$$

Considering nonoverlapping contiguous data segments, the next STFT will be calculated at instant  $n + M$ , as follows:

$$\mathbf{STFT}_M(n + M) = \mathbf{W}_M \mathbf{x}(n + M).$$

The last STFT at instant  $n + N - M$ , (assuming that  $N/M$  is an integer) is:

$$\mathbf{STFT}_{N-M}(n + N - M) = \mathbf{W}_M \mathbf{x}(n + N - M).$$

Combining all STFT vectors in a single equation, we obtain:

$$\mathbf{STFT} = \mathbf{W} \mathbf{x}. \quad (11)$$

In order to avoid confusion with notations, we will emphasize again that  $STFT(n, k)$  represents a scalar STFT value at a given time  $n$  and frequency  $k$ , while the boldface notation  $\mathbf{STFT}_M(n)$  with one argument and one index represents the vector of STFT values (at  $M$  frequencies for a given instant  $n$ ). Finally, boldface notation  $\mathbf{STFT}$  without arguments denotes vector of STFT values for all frequencies  $k$  and all instants  $n$ .

The STFT vector is

$$\mathbf{STFT} = \begin{bmatrix} \mathbf{STFT}_M(0) \\ \mathbf{STFT}_M(M) \\ \vdots \\ \mathbf{STFT}_M(N-M) \end{bmatrix}$$

and the  $N \times N$  coefficients matrix is formed as:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_M & \mathbf{0}_M & \cdots & \mathbf{0}_M \\ \mathbf{0}_M & \mathbf{W}_M & \cdots & \mathbf{0}_M \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_M & \mathbf{0}_M & \cdots & \mathbf{W}_M \end{bmatrix}, \quad (12)$$

where  $\mathbf{0}_M$  is a  $M \times M$  matrix with all 0 elements. The vector  $[\mathbf{x}(0), \mathbf{x}(M), \dots, \mathbf{x}(N-M)]^T$  is the signal vector  $\mathbf{x}$ , since:

$$\begin{aligned} \mathbf{x} &= [\mathbf{x}(0)^T, \mathbf{x}(M)^T, \dots, \mathbf{x}(N-M)^T]^T \\ &= [x(0), x(1), \dots, x(N-1)]^T. \end{aligned} \quad (13)$$

Expressing the above vector in the Fourier domain,

$$\mathbf{x} = \mathbf{W}_N^{-1} \mathbf{X}, \quad (14)$$

where  $\mathbf{W}_N^{-1}$  denotes the inverse DFT matrix of the dimension  $N \times N$  and  $\mathbf{X}$  is the DFT vector, we have:

$$\begin{bmatrix} \mathbf{STFT}_M(0) \\ \mathbf{STFT}_M(M) \\ \vdots \\ \mathbf{STFT}_M(N-M) \end{bmatrix} = \mathbf{W} \mathbf{W}_N^{-1} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}. \quad (15)$$

Accordingly, the relation between the STFT and DFT values can be written as follows:

$$\mathbf{STFT} = \mathbf{A}_{FULL} \mathbf{X}. \quad (16)$$

Matrix  $\mathbf{A}_{FULL} = \mathbf{W} \mathbf{W}_N^{-1}$  maps the global frequency information in  $\mathbf{X}$  into local information in  $\mathbf{STFT}$ . By removing a set of TF points using L-statistics, as discussed in Section II, only few elements in the observation vector  $\mathbf{STFT}$  remain. This is accomplished as follows. For each frequency  $k$ , a set of the STFT values in time is formed as:

$$\mathbf{S}_k(n) = \{STFT(n, k), n = 0, \dots, N-1\}. \quad (17)$$

After sorting the elements of  $\mathbf{S}_k(n)$ , we obtain the new ordered set of elements:

$$\Psi_k(n) \in \mathbf{S}_k(n),$$

such that

$$|\Psi_k(0)| \leq \dots \leq |\Psi_k(N-1)|. \quad (18)$$

A percentage  $Q$  of the high and low value elements are removed from consideration. As previously explained, these values capture most of the interference TF samples. The remaining STFT values belong to the desired signal.

Denote the vector of available STFT values by  $\mathbf{STFT}_{CS}$ . The corresponding matrix  $\mathbf{A}_{CS}$ , relating the sparse DFT vector  $\mathbf{X}$  to  $\mathbf{STFT}_{CS}$ , is formed by omitting the rows corresponding to the removed STFT values. Each row corresponds to one time and frequency point  $(n, k)$ . We maintain that the reduced observations, the sparse DFT domain, and the linear relationship provide the necessary ground of a CS problem. The goal is to reconstruct the original sparse stationary signal, since it produces the best concentrated DFT  $X(k)$ . Therefore, the corresponding minimization problem can be defined as follows:

$$\begin{aligned} \min \|\mathbf{X}\|_1 &= \min \sum_{k=0}^{N-1} |X(k)| \text{ subject to} \\ \mathbf{STFT}_{CS} &= \mathbf{A}_{CS}\mathbf{X}. \end{aligned} \quad (19)$$

Thus, based on the values of  $\mathbf{STFT}_{CS}$ , the missing values can be reconstructed such as to provide minimal  $\sum_{k=0}^{N-1} |X(k)|$ . This is a well known CS problem formulation which can be solved, if the recovery conditions are met, by using, for example, the primal-dual CS algorithm, [1], [2]. It will be shown that the amount of discarded samples allowing recovery is not very restrictive for performance, as long as the corrupted samples are discarded and sufficient information about the desired signal remains.

Norm  $\ell_1$  is used in minimization of (19) as a simpler form for the realization than the  $\ell_0$  form that would simply count the number of nonzero components. The norms  $\ell_p$  with  $0 \leq p \leq 1$  are used for the optimization of the time-frequency representation parameters in [12].

#### A. Time and Frequency Varying Windows

A relation similar to (16) can be obtained if the STFT has a time-varying window length. Assume that we use a window length  $M_0$  for the instant  $n = 0$  and calculate  $\mathbf{STFT}_{M_0}(0) = \mathbf{W}_{M_0}\mathbf{x}(0)$ . Then, we skip  $M_0$  signal samples. At  $n = M_0$ , a window of  $M_1$  samples is used to calculate  $\mathbf{STFT}_{M_1}(M_0) = \mathbf{W}_{M_1}\mathbf{x}(M_0)$ , and so on, until the last one  $\mathbf{STFT}_{M_L}(M_L) = \mathbf{W}_{M_L}\mathbf{x}(N - M_L)$  is obtained. Assuming that  $M_0 + M_1 + \dots + M_L = N$ , we can write:

$$\begin{bmatrix} \mathbf{STFT}_{M_0}(0) \\ \mathbf{STFT}_{M_1}(M_0) \\ \vdots \\ \mathbf{STFT}_{M_L}(N - M_L) \end{bmatrix} = \mathbf{A}_{FULL} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \quad (20)$$

or  $\mathbf{STFT} = \mathbf{A}_{FULL}\mathbf{X}$ , where

$$\mathbf{A}_{FULL} = \begin{bmatrix} \mathbf{W}_{M_0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{M_1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{M_L} \end{bmatrix} \mathbf{W}_N^{-1}.$$

Again for the available STFT values of  $\mathbf{STFT}$  denoted by  $\mathbf{STFT}_{CS}$ , the CS matrix  $\mathbf{A}_{CS}$  is defined by using the corresponding rows of  $\mathbf{A}_{FULL}$ . The problem has an identical form to (19), and can be solved using  $\ell_1$  norm signal reconstruction techniques.

As an example of time-varying windows, consider the form that assumes very short windows at early instants and then increases window width [30], [31]. It means that for "early samples" we have the best time-resolution, without frequency resolution. This is achieved with a one sample window. That is, at  $n = 0$ , the best time resolution is obtained with  $M_0 = 1$ ,  $\mathbf{STFT}_1(0) = \mathbf{W}_1\mathbf{x}(0) = x(0)$ . For an even number  $N$ , the same should be repeated for the next sample,  $n = 1$ , i.e.,  $\mathbf{STFT}_1(1) = \mathbf{W}_1\mathbf{x}(1) = x(1)$ . At the time instant  $n = 2$ , we decrease the time resolution and increase the frequency resolution by a factor of 2. It is achieved by using a two sample window in the STFT,  $M_2 = 2$ , so we have  $\mathbf{STFT}_2(2) = \mathbf{W}_2\mathbf{x}(2)$ . For the next instant,  $n = 4$ , we again increase the frequency resolution and decrease the time resolution by using a window of the length  $M_4 = 4$ ,  $\mathbf{STFT}_4(4) = \mathbf{W}_4\mathbf{x}(4)$ . Continuing in this way up to  $M_L$  we obtain

$$\begin{bmatrix} \mathbf{STFT}_1(0) \\ \mathbf{STFT}_1(1) \\ \mathbf{STFT}_2(2) \\ \vdots \\ \mathbf{STFT}_{M_L}(N - 2M_L) \\ \mathbf{STFT}_{M_L}(N - M_L) \end{bmatrix} = \mathbf{W}_D \mathbf{W}_N^{-1} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

where

$$\mathbf{W}_D = \begin{bmatrix} 1 & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ 0 & 1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{W}_4 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{M_L} \end{bmatrix}.$$

The CS matrix for this transform is obtained by omitting the rows corresponding to the unavailable (corrupted) STFT values from the full transformation matrix  $\mathbf{A}_{FULL}$ . The problem, again reduces to (19).

If we consider a signal with  $N$  samples, then its TF plane can be split into a large number of different tiles for the nonoverlapping STFT calculations. Let  $F(N)$  denotes the number of ways (patterns) an  $N \times N$  plane can be tiled. Then it can be readily shown that using diadic windows the approximative formula for  $F(N)$  is, [24]:

$$F(N) \approx [1.0366(1.7664)^{N-1}], \quad (21)$$

where  $[.]$  stands for an integer part of the argument. For example, for  $N = 16$  we have 5272 different ways to split TF

plane into nonoverlapping time-varying TF tiles. For general time-varying windows  $F(N) = 2^{N-1}$ . The STFT may use frequency-varying windows as well. Then, for a given frequency the constant window widths in time are used. Combining time-varying and frequency-varying windows we get hybrid TF-varying tiling. For any pattern that can satisfy recovery conditions, a reduced STFT observation vector can be defined by eliminating some tiles. The problem again lends itself to formulation (19).

In the cases with time and frequency varying windows the frequency grid changes as well. By appropriate interpolations in time and frequency the data can be adjusted for presentation in a constant TF grid. In the case of strong nonstationary disturbances, instead the L-statistics application for a given frequency, it is easier to apply the L-statistics on the whole analyzed set in order to remove the largest STFT values, corrupted by a strong disturbance. The sparse signal components are then recovered from the remaining/available STFT values, provided that they contain sufficient information for recovery. A special case of this kind of analysis, with  $M = 1$ , is presented in [11] for a separation of the stationary sparse part of the signal from a strong impulsive noise disturbance.

We may also use the STFT form dual to (8). It can be calculated using the signal's DFT:

$$\begin{aligned} STFT^F(n, k) &= \frac{1}{M} \sum_{i=0}^{M-1} X(k+i) e^{j2\pi in/M}, \\ \mathbf{STFT}_M^F(k) &= \mathbf{W}_M^{-1} \mathbf{X}(k). \end{aligned} \quad (22)$$

The forms that correspond to frequency-varying windows, dual to the ones for the time-varying windows (20), can be defined:

$$\mathbf{STFT}^F = \begin{bmatrix} \mathbf{W}_{M_0}^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{M_1}^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_{M_L}^{-1} \end{bmatrix} \mathbf{X}, \quad (23)$$

where  $\mathbf{X} = [X(0), X(1), \dots, X(N-1)]^T$  is the DFT vector and  $M_L$  are large enough that information about  $X(k)$  is contained in sufficient number of observations, after some STFT values are removed. The CS form is obtained by using remaining STFT values along with the corresponding rows in the transformation matrix.

### B. Overlapping STFT Case With Constant Windows

In general, for a signal  $x(n)$  of duration  $N$  we may use an arbitrary window  $w(m)$  of an arbitrary duration. The time step  $M_s$  in the STFT calculation may also be arbitrary. The STFT in terms of the signal's DFT reads:

$$\begin{aligned} STFT(n, k) &= \sum_{m=0}^{M-1} w(m) \left[ \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{j2\pi p(m+n)/N} \right] e^{-j2\pi mk/M} \\ &= \frac{1}{N} \sum_{p=0}^{N-1} X(p) e^{j2\pi pn/N} W(k - pM/N) = \mathbf{A}_n \mathbf{X}, \end{aligned} \quad (24)$$

where

$$W(k) = \sum_{m=0}^{N-1} w(m) e^{-j2\pi mk/N}$$

and the matrix  $\mathbf{A}_n$  coefficients are defined by:

$$A_n(k, p) = \frac{1}{N} W(k - pM/N) e^{j2\pi pn/N}.$$

A matrix notation of the STFT, with a step  $M_s$  in time, is:

$$\begin{bmatrix} \mathbf{STFT}_{M_s}(0) \\ \mathbf{STFT}_{M_s}(M_s) \\ \vdots \\ \mathbf{STFT}_{M_s}(N - M_s) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_{M_s} \\ \vdots \\ \mathbf{A}_{N-M_s} \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}. \quad (25)$$

Note that each STFT is actually a weighted linear combination of signal samples. These combinations are linearly independent only in the nonoverlapping STFT case. In the case of overlapping STFT the same signal samples are used to calculate several STFT values. Thus, they are not independent. For instance, using a step  $M/4$  instead of  $M$ , we get  $4N$  STFT values, where the same samples are involved in four (overlapping) different STFTs. This further means that if just one sample is corrupted by a disturbance, several TF values will be removed in a row.

In order to eliminate the nonstationary part of the STFT, containing  $x_{ns}(n)$ , we remove  $Q\%$  of the STFT values, for each frequency. Then we proceed to try to solve the CS problem similar to (19) with corresponding  $\mathbf{A}_{CS}$  and  $\mathbf{STFT}_{CS}$ .

If we want to keep the frequency grid in the STFT as in the original DFT of the signal then the windows should be zero-padded up to  $N$ , with  $M = N$  in (24). The zero-padding will provide that for each signal component with a constant frequency  $k_{0i}$  on the DFT grid, there is a frequency-direction line in the time-frequency plane where its STFT values have the same and constant phase for all considered instants. Calculating the STFT in this way, will increase the number of equations. However, an efficient CS solution may be obtained using the orthogonal matching pursuit methods with the initial recovered signal values obtained by setting the missing STFT values to zero, [15], [34].

As an example of overlapping windows, consider one practically interesting and commonly used case of the overlapping STFT calculation with a Hanning window of length  $M$  and time step  $M/2$ . Here, we will write a direct matrix formulation by splitting the STFT into two nonoverlapping forms. The STFT with a Hanning (or any other window) may be written in a matrix form as follows:

$$\begin{aligned} STFT(n, k) &= \sum_{m=0}^{M-1} w(m) x(n+m) e^{-j2\pi mk/M} \\ \mathbf{STFT}_M(n) &= \mathbf{W}_M \mathbf{H}_M \mathbf{x}(n), \end{aligned} \quad (26)$$

where  $\mathbf{H}_M$  is a diagonal  $M \times M$  matrix with the window values on the diagonal,  $H(i, i) = w(i)$ ,  $i = 0, 1, \dots, M-1$ . In ad-

dition, we introduce the first STFT with a half of the window length, as:

$$\begin{aligned} STFT(-M/2, k) &= \sum_{m=M/2}^{M-1} w(m)x(-M/2+m)e^{-j2\pi mk/M}, \\ STFT_{1/2}(0, k) &= \sum_{m=0}^{M/2-1} w(M/2+m)x(m)e^{-j2\pi mk/(M/2)} \\ \mathbf{STFT}_{M/2}(0) &= \mathbf{W}_{M/2} \mathbf{H}_{M/2}^+ \mathbf{x}_{M/2}(0), \end{aligned} \quad (27)$$

with  $STFT_{1/2}(0, k) = (-1)^k STFT(-M/2, 2k)$  and  $x(n) = 0$ , for  $n < 0$ . Here,  $\mathbf{H}_{M/2}^+$  is a diagonal  $M/2 \times M/2$  matrix,

$$\begin{aligned} H^+(i, i) &= w(i + M/2), i = 0, 1, \dots, M/2 - 1, \\ &\text{and} \\ \mathbf{x}_{M/2}(0) &= [x(0)x(1) \dots x(M/2 - 1)]^T. \end{aligned}$$

In the same way, the last STFT, with half of the window, is defined as:

$$\begin{aligned} STFT_{1/2}(N - M/2, k) \\ &= \sum_{m=0}^{M/2-1} w(m)x(N - M/2 + m)e^{-j2\pi mk/(M/2)} \\ \mathbf{STFT}_{M/2}(N - M/2) &= \mathbf{W}_{M/2} \mathbf{H}_{M/2}^- \mathbf{x}_{M/2}(N - M/2), \end{aligned}$$

where  $\mathbf{H}_{M/2}^-$  denotes a diagonal  $M/2 \times M/2$  matrix:  $H^-(i, i) = w(i)$ ,  $i = 0, 1, \dots, M/2 - 1$ , while

$$\mathbf{x}_{M/2}(N - M/2) = [x(N - M/2) \dots x(N - 2)x(N - 1)]^T.$$

Now it is possible to split the calculation into two sets. The first nonoverlapping set is:

$$\mathbf{STFT} = \mathbf{W}\mathbf{H}\mathbf{x}, \quad (28)$$

where  $\mathbf{W}$  and  $\mathbf{H}$  denote  $N \times N$  matrices, extended from  $M \times M$  transformation and windows matrices  $\mathbf{W}_M$  and  $\mathbf{H}_M$ , respectively, as in (11)–(13). The other, nonoverlapping, set is:

$$\mathbf{STFT}^{1/2} = \mathbf{W}^{1/2} \mathbf{H}^{1/2} \mathbf{x}, \quad (29)$$

where  $\mathbf{W}^{1/2}$  and  $\mathbf{H}^{1/2}$  are just notations for the matrices

$$\mathbf{W}^{1/2} = \begin{bmatrix} \mathbf{W}_{M/2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_M & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{W}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{W}_{M/2} \end{bmatrix}$$

and

$$\mathbf{H}^{1/2} = \begin{bmatrix} \mathbf{H}_{M/2}^+ & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_M & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{M/2}^- \end{bmatrix}.$$

The first and the last blocks are made by using half size matrices  $M/2 \times M/2$  and  $\mathbf{STFT}^{1/2}$  denotes corresponding STFT vector

$$\mathbf{STFT}^{1/2} = \begin{bmatrix} \mathbf{STFT}_{M/2}(0) \\ \mathbf{STFT}_M(M/2) \\ \vdots \\ \mathbf{STFT}_M(N - 3M/2) \\ \mathbf{STFT}_{M/2}(N - M/2) \end{bmatrix}.$$

The inversion (important to get a result free of the block effects) follows from (28), (29):

$$\begin{aligned} \mathbf{W}^{-1} \mathbf{STFT} + (\mathbf{W}^{1/2})^{-1} \mathbf{STFT}^{1/2} &= \\ \mathbf{H}\mathbf{x} + \mathbf{H}^{1/2} \mathbf{x} &= (\mathbf{H} + \mathbf{H}^{1/2}) \mathbf{x} = \mathbf{I}_N \mathbf{x} = \mathbf{x}. \end{aligned} \quad (30)$$

Similar inversion relation holds for the Hamming and triangular windows. It can be extended, in straightforward way, to other time steps of  $M/4$ ,  $M/8$ ,  $M/16$ ,  $\dots$ , 1, according to the overlap and add method. For the time step of 1, the inversion relation holds for any window form.

The CS convenient formulation of the overlapping case is:

$$\begin{bmatrix} \mathbf{STFT} \\ \mathbf{STFT}^{1/2} \end{bmatrix} = \begin{bmatrix} \mathbf{W}\mathbf{H} \\ \mathbf{W}^{1/2} \mathbf{H}^{1/2} \end{bmatrix} \mathbf{W}_N^{-1} \mathbf{x}, \quad (31)$$

where  $\mathbf{x} = \mathbf{W}_N^{-1} \mathbf{X}$  is used. The CS matrix is obtained by omitting interference STFT values. In general, the values in  $\mathbf{STFT}$  and  $\mathbf{STFT}^{1/2}$  are related to neighboring time instants. However, it should be noted that there is a time instant in the middle of each window that belongs only to one of these STFTs, calculated for various instants. Thus, theoretically, in this case it is possible to get all independent STFT values. For example, this is the case of a random signal  $x(n)$  with white (statistically independent) pulses at  $kM/2$ ,  $k = 0, 1, 2, \dots$ . In general, a disturbance can be large in one STFT, at one frequency, while in the neighboring STFT it could be significantly reduced, at the same frequency. By using overlapped STFTs, we may improve time resolution of the separation.

#### IV. NUMERICAL EXAMPLES

1) *Example 1:* Consider a desired signal that consists of four stationary sinusoids:

$$\begin{aligned} x(n) &= e^{j256\pi n/N} + 1.2e^{-j256\pi n/N + j\pi/8} \\ &\quad + 0.7e^{j512\pi n/N + j\pi/4} + e^{-j512\pi n/N - j\pi/3}, \end{aligned} \quad (32)$$

in nonstationary disturbances. The disturbances are of the form of short duration modulated signals (some of them are at the same frequencies as the stationary sinusoids):

$$x_{ch}(n) = \sum_{i=1}^{I_c} A_i e^{j\omega_i n} e^{-(n-n_i)^2/d_i^2}, \quad (33)$$

including four components whose changes of frequencies  $\omega_i$  follow a sinusoidal law. The data in time domain is shown in Fig. 2. The STFT is calculated for  $N = 1024$  and  $M = 32$ . The spectrogram of the data is presented in Fig. 3(a). After the L-statistics is performed on the sorted STFT values (Fig. 3(b)) and 50% of the largest values are removed along with 10% of

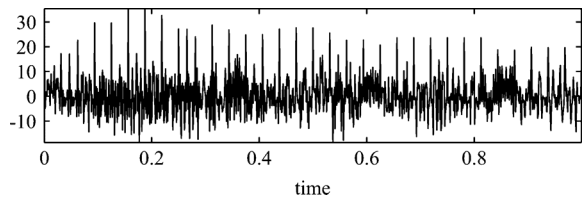


Fig. 2. Signal composed of a stationary part (sparse in frequency domain) and nonstationary part.

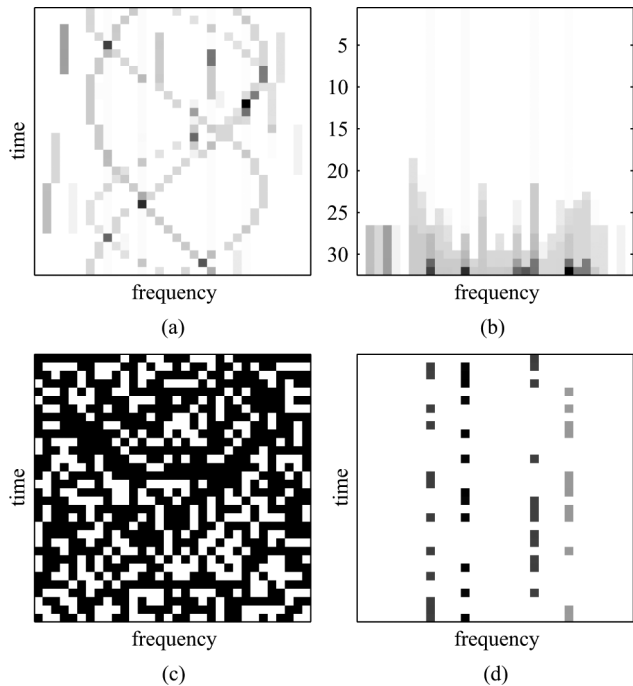


Fig. 3. a) STFT of the composite signal (spectrogram), b) Its sorted values, c) CS mask corresponding to the L-statistics based STFT values, d) STFT values (spectrogram) that remain after applying the L-statistics on the absolute values of the STFT.

the smallest values, the CS form of the STFT is obtained. The CS mask corresponding to the L-statistics based STFT values is shown in Fig. 3(c). The CS spectrogram values (that remain after the L-statistics based removal) are shown in Fig. 3(d).

The signal reconstruction is performed based on the STFT values from Fig. 3(d). Compared to the DFT of the data in Fig. 4(a), the reconstructed DFT shown in Fig. 4(b) is equal to the original DFT, preserving amplitude and phase.

The proposed method is compared with the results produced by an ideal case of notch filter (its inverse form), with the assumption that the signal's frequencies we are looking for are a priori known. The filter's frequency response for sinusoidal signal is  $H(k) = \delta(k - k_0)$  in the ideal case. The response includes all values along the considered frequency, producing wrong amplitude estimate as follows:  $Y_{notch}(k) = H(k)X(k) = 1690\delta(k - 257)$ , as in Fig. 4(a), instead of the true one which is in the considered example:  $X_{stat}(257) = 1024$ , Fig. 4(b). In other words, the starting instant is the Fourier transform in Fig. 4(a), which is quite affected by the disturbances and thus even in the considered idealized case, results with a high error are obtained. Note that zero-frequency is at  $k = N/2 + 1$ .

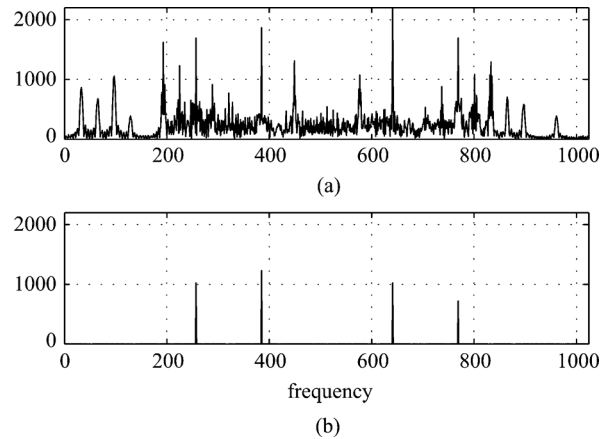


Fig. 4. Fourier transform of the original composite signal (a). The reconstructed Fourier transform by using the CS values of the STFT, corresponding to sparse part of the composite signal (b).

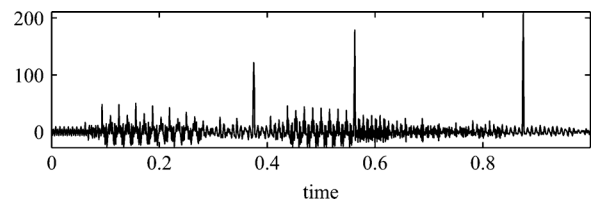


Fig. 5. Signal composed of a sparse part and nonstationary part, with additional transient impulses in time, including three strong pulses.

2) *Example 2*: The desired sparse signal from the previous example is considered, with short duration pulses and strong transient signals as disturbances. The data is depicted in Fig. 5. This case can be presented within the CS framework using the TF domain. The STFT of the original signal (spectrogram) is shown in Fig. 6(a). After the L-statistics approach is performed on the sorted STFT values with 60% of the largest values and 10% of the lowest values being removed ( $Q = 70\%$ ), the result is presented in Fig. 6(b), which amounts to the CS form of the STFT. The omitted STFT values are indicated in Fig. 6(c). The reconstruction is performed based on the CS values of the STFT given in Fig. 6(d).

The DFT of the sparse part of signal is obtained, Fig. 7 (c), with both the amplitude and the phase preserved. The DFT of original signal is presented in Fig. 7 (a). The DFT obtained by using the  $\ell_2$  norm, in lieu of the  $\ell_1$  norm when solving (19) is shown in Fig. 7(b).

In the sequel, we use the same example to demonstrate a low sensitivity of the proposed algorithm to the choice of  $Q$ . Namely, a very wide range of  $Q$  can be successfully used in the applications. Thus, the CS algorithms will provide efficient signal reconstruction as far as most of nonstationary disturbances are removed. However, if we omit more than 85%-90% of values, we do not have enough information for signal reconstruction. The influence of various values of  $Q$  has been measured by the MSE between the desired and reconstructed Fourier transform. The results are shown in Fig. 8 (for the proposed  $\ell_1$ -based CS and for the  $\ell_2$  reconstruction). Observe that the MSE decreases as the amount of removed disturbances increases, being almost negligible between  $Q = 50\%$  and



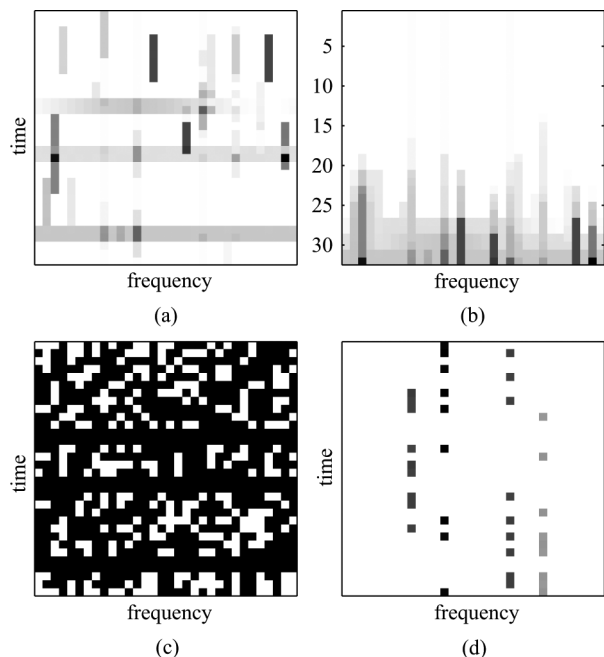


Fig. 6. (a) The STFT of the composite signal with impulse disturbances in time domain (spectrogram). (b) Its sorted values. (c) The CS mask corresponding to the L-statistics based STFT values (d) The STFT values (spectrogram) that remain after applying the CS mask on the absolute values of the STFT.

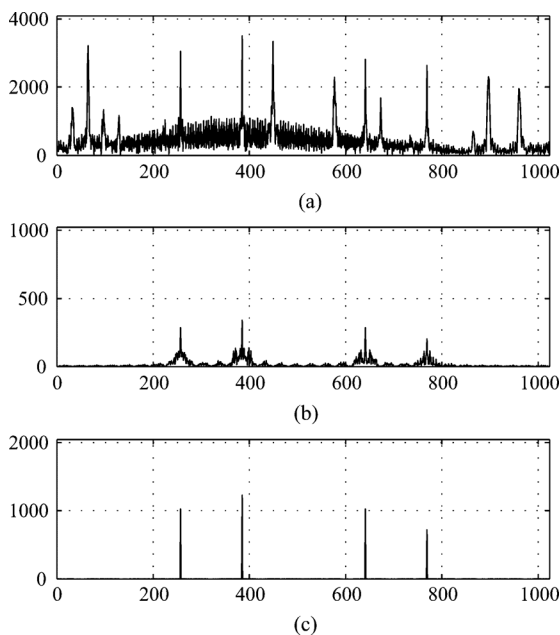


Fig. 7. (a) The Fourier transform (DFT) of: The original signal. (b) The signal reconstructed by using the  $\ell_2$  norm, on the set of data after L-statistics is applied in time-frequency domain. (c) The reconstructed Fourier transform by using the proposed method with the CS values of the STFT.

$Q = 80\%$ . Namely, in this range of  $Q$  the disturbances are negligible, while we still have enough useful information for the CS based reconstruction. Further increase of  $Q$  ( $Q \geq 80\%$ ) increases the MSE due to the lack of signal information required for the efficient CS algorithm application. Thus, we may conclude that a good choice will be the highest value of  $Q$  for which the CS reconstruction still can be used. Robust transforms, calculated on the whole set of data (including the

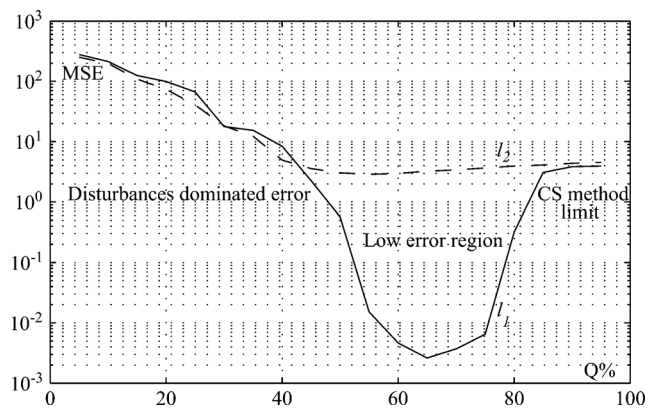


Fig. 8. MSE between the desired and reconstructed FT obtained using the proposed  $\ell_1$ -based approach (solid line) and  $\ell_2$ -based approach (tick line), for different values of parameter  $Q$ . The MSE value normalized by the number of samples  $N$  is presented.

application of the  $\ell_1$  norm in such analysis) are studied in [22], [32].

The result with  $\ell_2$  norm is obtained as a least squared solution of the previous minimization problem (19). The  $\ell_2$  norm on the data set that remains after the L-statistics is applied. It can be written in the form:

$$\begin{aligned} \text{STFT}_{CS} &= \mathbf{A}_{CS} \mathbf{X} \text{ or} \\ \mathbf{A}_{CS}^* \text{STFT}_{CS} &= \mathbf{A}_{CS}^* \mathbf{A}_{CS} \mathbf{X} \end{aligned}$$

with

$$\mathbf{X} = (\mathbf{A}_{CS}^* \mathbf{A}_{CS})^{-1} \mathbf{A}_{CS}^* \text{STFT}_{CS},$$

where  $*$  denotes Hermitian transpose. It is clear that the MSE is significantly decreased when using the CS, Fig. 7. The results obtained with  $\ell_2$  norm could be used as an initial representation in the orthogonal matching pursuit CS algorithms. The results from [15] could also be used as the initial representation in this sense.

In this simulation example, a simple a posteriori check of the accuracy was possible since the exact signal values are known. The exact recovery of a sparse signal is considered using the restricted isometry property (RIP), [33]. The sufficient condition for recovery is defined by using the squared  $\ell_2$  norms (energies) of the sparse signal and the available measurements. In general, the exact recovery depends on the signal and samples (measurements) positions. For example, a single discrete sinusoid can exactly be recovered using just a few of its samples, with a high probability. However the exact recovery is not guaranteed, even for a very large number of available samples. For example, with 512 out of 1024 samples (taking every other sample), we will not be able even to detect a high frequency discrete sinusoid. A simple stochastic analysis of the results can be performed similarly as in [34].

3) *Example 3:* In general, if a disturbance component covers all considered time interval, and if it is not quasi-stationary, then an appropriate window function should be used to localize the disturbance in frequency. It will provide that (6) is satisfied at least for some instants. The case with a Hanning localization window, in the overlapping STFT calculation, is considered in

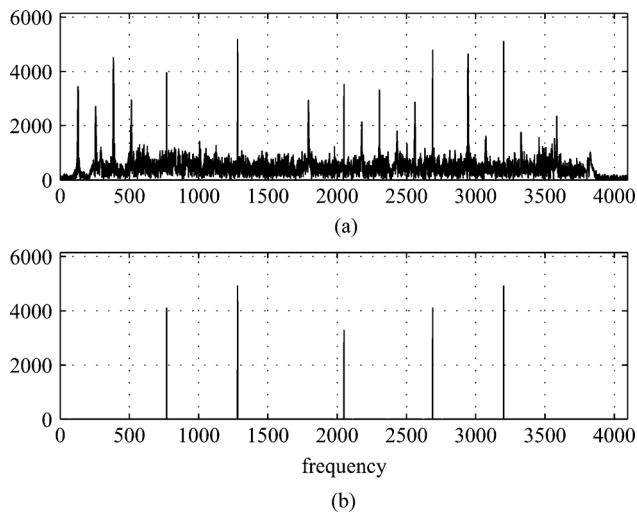


Fig. 9. The Fourier transform (DFT) of: (a) The original composite signal. (b) The reconstructed Fourier transform by using the proposed method with the CS values of the STFT.

this example. The data and the reconstruction results are presented in Fig. 1 (used as an illustration for the paper motivation). The sparse signal is given in the form:

$$x_{sp}(n) = e^{j640\pi n/N} + 0.8e^{-j\pi/8} + 1.2e^{-j768\pi n/N + j\pi/4} + 1.2e^{j1152\pi n/N + j\pi/4} + e^{-j120\pi n/N + j\pi/4}$$

with  $N = 2048$ . The disturbance consists of four sinusoidally modulated signals and 22 shorter duration signals of the form  $A_i \exp(j\omega_i n/N + j\phi_i) \exp(-(n - n_{0i})/d_i)^p$ . Different amplitudes are assumed for 22 components all in the range between  $A_i = 1$  and  $A_i = 5.5$ , with durations (defined by  $d_i$ ). Some of the disturbance terms appear at the same frequency  $\omega_i$  as the signal (stationary) components. Additive noise with standard deviation  $\sigma = 0.25$  is present as well. The STFT is calculated by using Hanning window of the width  $M = 64$ . A Hanning window, zero-padded up to the full signal length, could also be used to provide a fine frequency grid in the time-frequency analysis, [15]. The STFTs are calculated with the step  $M_s = 32$ , i.e., with a half of the overlapping window. The STFT absolute value is presented in Fig. 1(a). The values are sorted and the L-statistics is performed (Fig. 1(b)) with  $Q = 70\%$  discarded samples. The CS mask is presented in Fig. 1(c), while the result of applying the CS method to the STFT (with CS mask) are shown in Fig. 9.

## V. CONCLUSION

The problem of recovering stationary narrowband signals contaminated by strong nonstationary signals was addressed. The reduced observations are computed at selected TF points. This selection is provided through the L-statistics which discards the TF points believed to belong to the interference or the undesired signal components. With the desired signal of sparse frequency domain representation, the problem can be cast as compressive sensing aiming at the recovery of narrowband signals in interference using  $\ell_1$  reconstruction algorithms. The matrix relating the sparse DFT vector of the desired signal to the observation vectors of selected STFT points, was defined

for both overlapping and nonoverlapping data segments. Generalizations to the time-varying and frequency-varying windows were provided. The proposed technique was successfully applied for the reconstruction of multiple sinusoids corrupted by a complex nonstationary disturbance, including impulsive noise.

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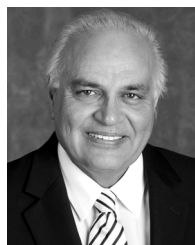
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