

Instantaneous Frequency in Time-Frequency Analysis: Concept and Estimation

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Abstract

The instantaneous frequency (IF) is very important feature of nonstationary signals in numerous applications. The first overview of the concept and application of the IF estimators is presented in seminal papers by Boashash. Since then, significant knowledge has been gained about the performance of the IF estimators. This knowledge has been used not only for development of various IF estimators but also for introduction of novel time-frequency (TF) representations. The IF estimation in high noise environments has achieved significant benefits from these theoretical developments. In this paper, we review some of the most important developments in the last two decades related to the concept of the IF, performance analysis of IF estimators, and development of IF estimators for high noise environments.

Keywords: Instantaneous frequency, time-frequency signal analysis, parametric estimation, non-parametric estimation, robust estimation.

1. Introduction

Signals generated and sampled in time are often analyzed and processed in the frequency domain. Basic tool for mapping a signal from time into frequency domain is the Fourier transform (FT). The FT-based analysis is efficient when the frequency content of the analyzed signal does not change over time. However, in many applications, we deal with signals where important information about the physical process is conveyed within the time variations of the signal's spectral components. The time-frequency (TF) analysis provides efficient tools for analyzing such signals. One of the most important parameters in the TF analysis is

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the instantaneous frequency (IF). Its representation, estimation, and relation to the physical quantities represent key topics in the TF analysis [1–5]. One of the oldest applications of signals with time-varying frequency is in communications based on frequency modulation, where the information is conveyed through the frequency variations. In radar and sonar systems, information about position and/or velocity of the target is contained in the IF of the returned signal [4, 6]. At each instant, speech signal can be represented with several FM components whose IFs bear information important for speech analysis and recognition [7]. The motion parameters of objects in video-sequences can be embedded in the IF of corresponding signals [8]. In seismology, TF tools can be very helpful in representing seismic signals and extracting the IF [9]. The IF estimation can be used in hydrocarbon detection, since in a reservoir containing fluid, frequency of seismic waves decreases faster than in surrounding rocks. In biology, the tracking of some animal populations can be performed using the IF extracted from sounds produced by these animals [10]. Numerous applications of the IF are present in medicine. For example, in newborn’s electroencephalogram (EEG), the IF estimation can be used for seizure detection, modelling and classification [11].

The concept of the IF for monocomponent and multicomponents signals is reviewed in Section II, along with a model of an ideal TF mapping using the stationary phase method. The empirical mode decomposition is related to the IF in multicomponent signals. The IF has been used to introduce many TF representations, reviewed in Section III. Here, a model for the IF estimation analysis is also presented, including an adaptive method for the IF estimation defined to track the IF with a minimal possible estimation error. An overview of the TF representations performance analysis in the case of high noise is given in this section as well. The parametric and combined approaches in the IF estimation are presented in Section IV.

2. Definitions

Consider a complex sinusoid

$$x(t) = Ae^{j(\Omega_0 t + \phi_0)} = Ae^{j\phi(t)}, \quad (1)$$

where A , Ω_0 and ϕ_0 represent the constant amplitude, (angular) frequency and initial phase of $x(t)$, respectively. Its phase derivative,

$$\Omega(t) = \phi'(t) = \Omega_0, \quad (2)$$

is constant and equal to frequency. The frequency Ω_0 can be considered as the rate of change of the phase function $\phi(t)$. Signal (1) can be interpreted as a vector with constant amplitude A and angle $\phi(t)$, rotating

with a constant angular velocity Ω_0 , making $\Omega_0/(2\pi)$ cycles per second. Signal form $x(t) = Ae^{j\phi(t)}$ can be generalized for an arbitrary differentiable phase function $\phi(t)$. Then, in analogy with (2), we can still interpret the phase derivative as the instantaneous rate of change of the phase function (instantaneous angular velocity).

Consider now a signal

$$x(t) = A(t)e^{j\phi(t)}, \quad (3)$$

where $A(t)$ represents a slow time-varying amplitude. Within the frequency framework, we can interpret the phase derivative at $t = t_0$ as the frequency of signal that would behave as a complex sinusoid with constant frequency $\Omega = \phi'(t_0)$ and amplitude $A(t_0)$. In that sense, we may consider a sinusoid $e^{j\Omega t}$ that locally fits $x(t)$ around t_0 , i.e., that produces constant phase of the product $x(t)e^{-j\Omega t}$ at the considered instant t_0 and its vicinity. Then $\phi'(t_0) = \Omega$ holds. This consideration leads to the *method of stationary phase* which states that if the phase function $\phi(t)$ is monotonous and amplitude $A(t)$ is sufficiently smooth function [12, 13], then the FT of a signal $A(t)e^{j\phi(t)}$ can be approximated as

$$\int_{-\infty}^{\infty} A(t)e^{j\phi(t)}e^{-j\Omega t} dt \simeq A(t_0)e^{j\phi(t_0)}e^{-j\Omega t_0} \sqrt{\frac{2\pi j}{\phi''(t_0)}}, \quad (4)$$

where t_0 is the solution of $\phi'(t_0) = \Omega$.

The most significant contribution to the integral on the left side of (4) comes from the region where the phase of $\exp(j(\phi(t) - \Omega t))$ is constant, since the contribution of intervals with fast varying $\phi(t) - \Omega t$ averages to zero. In a small time region around t , the phase $\phi(t)$ behaves as Ωt . Thus, we may say that, for a particular instant t , the IF (rate of the phase change, $\phi'(t)$) of signal $A(t)e^{j\phi(t)}$ corresponds to frequency Ω . In this way, for each time instant t , we obtain the corresponding frequency $\Omega = \phi'(t)$, and map the signal from the time into frequency domain. This mapping can be written as a two dimensional TF function by using $\delta(\Omega - \phi'(t))$ and amplitudes according to (4), which is illustrated in Fig. 1.

The method of stationary phase has been used as a starting point in developing reassignment techniques. Namely, for each TF point (t, Ω) , the TF representation values can be reassigned towards local gravity center (point where they contribute the most) in order to approach ideal TF representation in terms of concentrating energy along the IF and group delay, as its dual notion [14, 15].

In practical applications it is quite common that the analyzed signal is composed of several signals of form (3). For example, radar signal can contain signals reflected from various targets or signals received

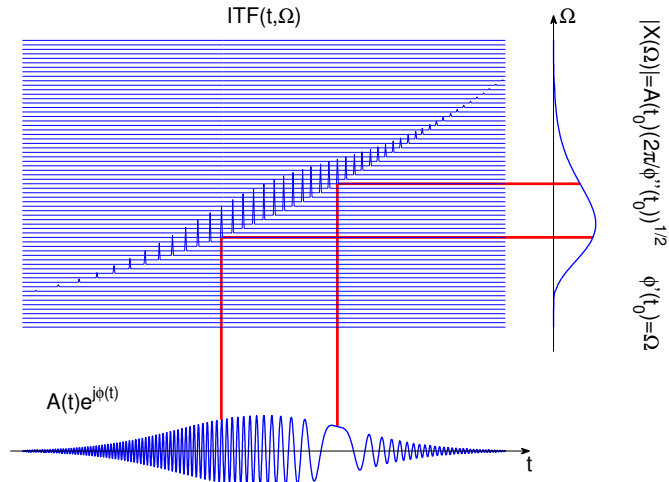


Figure 1: Illustration of the stationary phase method which relates a signal in time, at instant t , with the FT value at frequency $\Omega = \phi'(t)$ via the TF domain.

through multiple paths. Speech and sound signals are also very rich in components. Signals produced by vehicle engines contain components on several time-varying resonance frequencies. Such signals can be represented as

$$x(t) = \sum_{m=1}^M A_m(t) e^{j\phi_m(t)} = A(t) e^{j\phi(t)}, \quad (5)$$

where $A_m(t)$ are slow varying amplitudes. The resulting amplitude $A(t)$ variations can be of the same order as variations of the phase function. Here we may use estimators that produce a single resulting IF as $\Omega = \phi'(t)$. However, in the case of multicomponent signals (5), real physical quantities are usually not related to the single resulting IF; instead, they are related to the individual IFs $\Omega_m = \phi'_m(t)$, $m = 1, 2, \dots, M$, of the signal components. In many cases, signal components can be separated in time, frequency or TF domain so that the IF can be determined for each component. Various separation algorithms exist, while the TF-based ones are the most effective. Note that real valued signal $x(t) = 2 \sum_{m=1}^M A_m(t) \cos(\phi_m(t))$ could be considered as a special case of (5) with $x(t) = \sum_{m=1}^M A_m(t) e^{-j\phi_m(t)} + \sum_{m=1}^M A_m(t) e^{j\phi_m(t)}$. If functions $\phi_m(t)$, for $m = 1, 2, \dots, M$, are such that the FTs of $A_m(t) e^{j\phi_m(t)}$ equal zero at negative frequencies, the analytic part of $x(t)$ can be used, since the relation between a real valued signal and its analytic part is unique and simple in this case [12].

The vector representation of multicomponent FM signals can lead to another direction in the analysis.

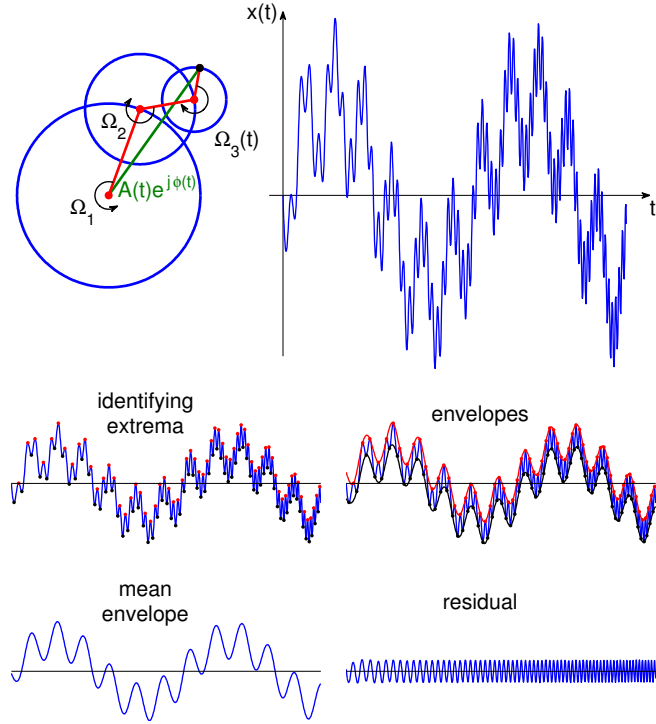


Figure 2: Vector illustration of a multicomponent signal and its decomposition.

If the rotation speeds of vectors are of different orders, we may imagine that our detection system is so inert that it cannot observe the fastest rotation. Then, we can detect the trajectory of the fastest rotating vector central point (e.g. by detecting local maxima and local minima caused by the fastest rotation). We remove this rotation, and continue in the same way with the next fastest rotation. This is the basic idea of the empirical mode decomposition (EMD) approach in the IF analysis [16]. The decomposition is illustrated on a three component signal $x(t) = A_1 e^{j\phi_1(t)} + A_2 e^{j\phi_2(t)} + A_3 e^{j\phi_3(t)}$, where $A_1 > A_2 > A_3$ and $|\phi'_1(t)| \ll |\phi'_2(t)| \ll |\phi'_3(t)|$. Imaginary part of $x(t)$ is used in decomposition presented in Fig. 2.

3. IF in TF Representations

The IF concept has played significant role in definition and analysis of TF representations. Some of the approaches inspired by the IF are presented in the previous section. The IF has been used to define numerous TF representations. The key idea behind these representations is to achieve high (full) concentration of the

signal energy along its IF

$$\begin{aligned} \text{ITF}(t, \Omega) &= 2\pi A^q(t) \delta(\Omega - \phi'(t)) \\ &= A^q(t) \int_{-\infty}^{\infty} e^{j\phi'(t)\tau} e^{-j\Omega\tau} d\tau, \end{aligned} \quad (6)$$

where q is the power of amplitude (depending on the representation order) and $\text{ITF}(t, \Omega)$ denotes the ideal TF representation. The whole signal generalized power $A^q(t)$ would be concentrated at a single point $\Omega = \phi'(t)$ for a given instant t . In that sense, TF representations could be defined as

$$\text{TF}(t, \Omega) = \int_{-\infty}^{\infty} \prod_{i=1}^p x^{b_i}(t + c_i\tau) e^{-j\Omega\tau} d\tau, \quad (7)$$

where p , b_i and c_i are constants. For signal $x(t) = A(t) \exp(j\phi(t))$, phase of the product in (7) is

$$\sum_{i=1}^p b_i \phi(t + c_i\tau) = \phi'(t)\tau + \Delta\phi(t, \tau). \quad (8)$$

The ideal representation (6) is achieved for the spread factor $\Delta\phi(t, \tau) = 0$. When $\Delta\phi(t, \tau) \neq 0$, the representation will be spread around the IF.

For example, we can easily derive the Wigner distribution (WD) following this analysis. For a linear FM signal, with $\phi(t) = at^2 + bt + c$, it is known that $\phi(t + \tau/2) - \phi(t - \tau/2) = \phi'(t)\tau$. Thus, the coefficients in (7) and (8) are $b_1 = 1$, $c_1 = 1/2$, $c_2 = -1/2$, and $b_2 = -1$ (negative b_2 means complex conjugate of signal), producing the WD, ideal representation of the linear FM signal.

In general, the phase functions $\phi(t + c_i\tau)$ are expanded in Taylor series around $\tau = 0$. The coefficients b_i and c_i in (8) should satisfy the following conditions:

- Sum of coefficients with $\phi(t)$ equals 0. This condition may be omitted if the absolute value of distribution is used.
- Sum of coefficients with $\phi'(t)\tau$ equals 1.
- Coefficients with $\phi^{(p)}(t)\tau^p$, up to the desired order, equal 0.

If we add the requirement that the TF representation is real-valued, then $c_{2i+1} = -c_{2i}$ and $b_{2i+1} = -b_{2i}$ hold. The fourth-order ($p = 4$) polynomial Wigner-Ville distribution (PWVD) ideally concentrates the signals with polynomial phase up to the fourth order [1]. Assuming integer signal powers in (7), we may

Table 1: Coefficients of some TF representations with their spread factors

Representation	Coefficients	Spread factor
Wigner distribution	$b_1 = -b_2 = 1$ $c_1 = -c_2 = 1/2$	$\phi'''(t)\tau^3/24 + \dots$
Rihaczek distribution	$b_1 = 1 \quad b_2 = -1$ $c_1 = 0 \quad c_2 = -1$	$\phi''(t)\tau^2/2 + \dots$
Polynomial Wigner-Ville distribution	$b_1 = -b_2 = 2$ $b_3 = -b_4 = -1$ $c_1 = -c_2 \simeq 0.675$ $c_3 = -c_4 \simeq 0.85$	$\phi^{(5)}(t)\tau^5/3 + \dots$
Complex- time distribution	$b_1 = -b_2 = 1$ $b_3 = -b_4 = j$ $c_1 = -c_2 = 1/4$ $c_3 = -c_4 = -j/4$	$\phi^{(5)}(t)\tau^5/(4^2 5!) + \dots$

show that the lowest possible powers are $b_1 = -b_2 = 2$ and $b_3 = -b_4 = -1$. Complex-time distributions may be derived using a similar analysis, allowing complex arguments [17]. The coefficients of some TF representations with their spread factors are presented in Table 1.

Similarly to (6), TF representations that are concentrated along the second phase derivative $\phi''(t)$, i.e., along the IF rate, can be introduced [18].

In the case of multicomponent signals, the ideal TF is of the form

$$\text{ITF}(t, \Omega) = \sum_{m=1}^M 2\pi A_m^q(t) \delta(\Omega - \phi'_m(t)).$$

Within the TF analysis framework, it implies cross-terms free (reduced) realization of TF representations. One efficient approach towards this aim is based on order-recursive realizations, starting from the STFT that is cross-terms free [19]. Another direction in the analysis of multicomponent signals is in dividing the TF domain into regions corresponding to individual components and applying the standard IF analysis within these regions. Since the ideal representation of an M -component signal contains only M nonzero values at the considered instant, it is sparse in frequency, and it can also be used as a starting point for definition and efficient implementation of algorithms for processing of signals that are sparse in the TF domain [20, 21].

3.1. IF Estimation Model

The IF estimation is crucial in numerous signal processing applications. Techniques based on the TF analysis represent the state-of-the-art in this area due to high accuracy and robustness to the noise influence. One of the basic properties of TF representations is that the IF can be obtained as [22]

$$\frac{\int_{-\infty}^{\infty} \Omega \text{TF}(t, \Omega) d\Omega}{\int_{-\infty}^{\infty} \text{TF}(t, \Omega) d\Omega} = \Omega(t) = \frac{d}{dt} \arg[x(t)] = \phi'(t). \quad (9)$$

This theoretically important relation is rarely used in the IF estimation since it is very sensitive to noise (multiplication by Ω increases noise at high frequencies). In the case of multicomponent signals, relation (9) will result in a single value that generally does not correspond to the IFs of the signal components. Instead, the nonparametric IF estimation based on the maxima position of the TF representation will be used as follows [1, 23, 24]:

$$\hat{\Omega}(t) = \arg \max_{\Omega} \text{TF}(t, \Omega). \quad (10)$$

The position of maximum $\Omega = \phi'(t)$ can be found as a solution of $\partial \text{TF}(t, \Omega) / \partial \Omega = 0$. Deviation of detected maxima positions from the signal's IF is caused by noise and spread factor $\Delta\phi(t, \tau)$. Including these

disturbances, $\partial\text{TF}(t, \Omega)/\partial\Omega$ can be modeled as

$$\frac{\partial}{\partial\Omega} \left(A^q(t) \int_{-\infty}^{\infty} w(\tau) e^{j(\phi'(t)\tau + \Delta\phi(t, \tau))} e^{-j\Omega\tau} d\tau + \Xi(t, \Omega) \right),$$

where $\Xi(t, \Omega)$ represents all terms in the TF representation influenced by noise. The TF maxima position is shifted from the true IF position by $\Delta\Omega$ due to higher order phase derivatives $\Delta\phi(t, \tau)$ and noisy terms $\Xi(t, \Omega)$. First we will present analysis assuming small disturbances in the sense that maximum position deviation remains within the auto-term. The width of the auto-term depends on the lag interval width and $\Delta\phi(t, \tau)$. For narrow lag intervals, $\Delta\phi(t, \tau) \rightarrow 0$. For small $\Delta\phi(t, \tau)$, it holds $\exp(j\Delta\phi(t, \tau)) \cong 1 + j\Delta\phi(t, \tau)$. The zero value of $\partial\text{TF}(t, \Omega)/\partial\Omega$ will be shifted from $\Omega = \phi'(t)$ to $\Omega = \phi'(t) + \Delta\Omega$. The error $\Delta\Omega$ can be analyzed by using a linear model of the TF derivative

$$\begin{aligned} \frac{\partial\text{TF}(t, \Omega)}{\partial\Omega} \Big|_{\Omega=\phi'(t)+\Delta\Omega} &= A^q(t) \int_{-\infty}^{\infty} w(\tau) \tau \Delta\phi(t, \tau) d\tau \\ + \frac{\partial\Xi(t, \Omega)}{\partial\Omega} \Big|_{\substack{\Omega=\phi'(t), \\ \Delta\phi(t, \tau)=0}} &+ \Delta\Omega \frac{\partial^2\text{TF}(t, \Omega)}{\partial\Omega^2} \Big|_{\substack{\Omega=\phi'(t), \\ \Delta\phi(t, \tau)=0, \\ \Xi(t, \Omega)=0}} \end{aligned}$$

since $\int_{-\infty}^{\infty} w(\tau) \tau d\tau = 0$. For the new maximum position of the disturbed TF, the IF estimate satisfies $\partial\text{TF}(t, \Omega)/\partial\Omega|_{\Omega=\phi'(t)+\Delta\Omega} = 0$. The approximative error in the IF estimation is

$$\Delta\Omega = \frac{A^q(t) \int_{-\infty}^{\infty} w(\tau) \tau \Delta\phi(t, \tau) d\tau + \frac{\partial\Xi(t, \Omega)}{\partial\Omega} \Big|_{\substack{\Omega=\phi'(t), \\ \Delta\phi(t, \tau)=0}}}{\frac{\partial^2\text{TF}(t, \Omega)}{\partial\Omega^2} \Big|_{\substack{\Omega=\phi'(t), \\ \Delta\phi(t, \tau)=0, \\ \Xi(t, \Omega)=0}}}.$$

For example, the bias and variance of the WD-based IF estimator with the Hann window of the width h are approximately

$$\text{E}\{\Delta\Omega\} = \text{Bias}(t, h) \cong \frac{1}{76} \Omega^{(2)}(t) h^2, \quad (11)$$

$$\text{Var}\{\Delta\Omega\} = \sigma^2(h) \cong 14 \frac{\sigma_{\varepsilon}^2}{A^2} \frac{1}{h^3}. \quad (12)$$

Obviously, the bias is an increasing function of h , while the variance is a decreasing one, pointing out that there is a trade-off in selection of the window width in the WD-based IF estimator. Similar bias and variance relations may be derived for other TF representations (spectrogram, PWVD, Cohen class of distributions) as functions depending on the kernel size, window width and distribution order, [25].

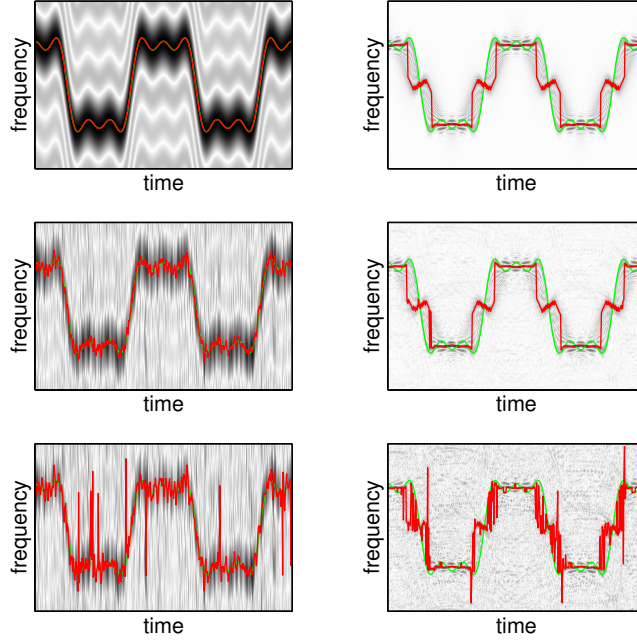


Figure 3: The IF estimation of noisy nonlinear FM signal by using the WD with a narrow window (left) and a wide window (right). Cases with negligible noise (top row), small noise (middle row), and high noise (bottom row) are presented. The true IF is presented by a green line, while the estimated IF is presented by a red line.

Figure 3 depicts the WD-based IF estimation of an FM signal corrupted by Gaussian noise. The top row in Fig. 3 compares the IF estimations obtained using a narrow window (left) and a wide window (right), when the noise is almost negligible. Narrow window based estimation follows the true IF, while the bias in the wide window based estimation is notable. Here the auto-term contains significant inner interferences caused by the spread factor. The IF estimates for the small noise case (SNR = 10dB) are presented in the middle row. Fluctuations within the auto-term caused by noise are significant in the narrow window case. The bottom row depicts the IF estimation in high noise, showing that some IF estimates completely miss the auto-term. The influence of high noise on the IF estimation will be considered later.

The MSE of the TF-based IF estimators can, in general, be written as the sum of the squared bias and variance as

$$\text{MSE}(t, h) = \sigma^2(h) + \text{Bias}^2(t, h) = V/h^m + B(t)h^n.$$

According to (11) and (12), for the WD we get $m = 3$, $n = 4$, $V = 14\sigma_\varepsilon^2/A^2$ and $B(t) = (\Omega^{(2)}(t)/76)^2$. Obviously, there is an optimal window $h(t) = h_{opt}(t)$ that follows from $\partial\text{MSE}(t, h)/\partial h = 0$. However, such

a window depends on signal itself since parameter $B(t)$ depends on the IF derivatives. For the estimation of the optimal window width $h_{opt}(t)$ without prior knowledge of parameter $B(t)$, an algorithm based on the intersection of the confidence intervals (ICI) is introduced [26], and briefly described here. Let us introduce a set H of window-width values $h_1 < h_2 < \dots < h_J$. Define the confidence intervals $D_s = [L_s, U_s]$ for $s = 1, 2, \dots, J$ of the estimates, with the following upper and lower bounds $L_s = \hat{\Omega}_s(t) - K\sigma(h_s)$, $U_s = \hat{\Omega}_s(t) + K\sigma(h_s)$, where $\hat{\Omega}_s(t)$ is an estimate of $\Omega(t)$, obtained from (10) using the window width h_s , and $\sigma(h_s)$ represents its standard deviation. For small window widths h_s , the bias of $\hat{\Omega}_s(t)$ is negligible, thus $\Omega(t) \in D_s$. Then, obviously, $D_{s-1} \cap D_s \neq \emptyset$, since at least the true value $\Omega(t)$, belongs to both confidence intervals. For wide windows the variance is small, but the bias is large. Clearly, a large enough s exists such that $D_s \cap D_{s+1} = \emptyset$ for a finite K and $\text{Bias}(t, h_s) \neq 0$. The idea behind the algorithm is that the optimal window width can be determined as highest h_s for which the successive confidence intervals D_{s-1} and D_s have at least one point in common. Intersection of the confidence intervals D_{s-1} and D_s

$$\left| \hat{\Omega}_{h_{s-1}}(t) - \hat{\Omega}_{h_s}(t) \right| \leq K[\sigma(h_{s-1}) + \sigma(h_s)], \quad (13)$$

works as an indicator of the event $h_s \sim h_{opt}$. The algorithm is not sensitive to the value of K , so $K \sim 3$ could be used [26]. There are many methods in literature for an efficient SNR estimation needed for calculation of $\sigma(h_s)$ [26–29].

The ICI algorithm performance is illustrated in Fig. 4. An FM signal corrupted by Gaussian white noise with SNR = 20dB is considered. Set of window widths $H = \{32, 38, 46, 54, 64, 76, 90, 108, 128, 152, 182, 216, 256\}$ is used. Narrow windows produce noisy estimates, whereas wider windows produce biased estimates. The ICI algorithm takes window width adaptively, producing accurate IF estimation. The algorithm uses the widest windows in constant and linear IF segments (no higher order phase derivatives), while narrower windows are used when the IF is not linear (significant higher order phase derivatives).

3.2. IF Estimation in High Noise

In high noise environments, there is a high probability that the TF values caused by noise exceed the TF values along the IF of the analyzed signal. The detected maximum then can be at arbitrary position, thus producing high estimation error (Fig. 3, bottom row). This effect is more emphasized for narrower windows used in calculation of the TF representation. Three strategies are in use for handling high noise influence on the IF estimation.

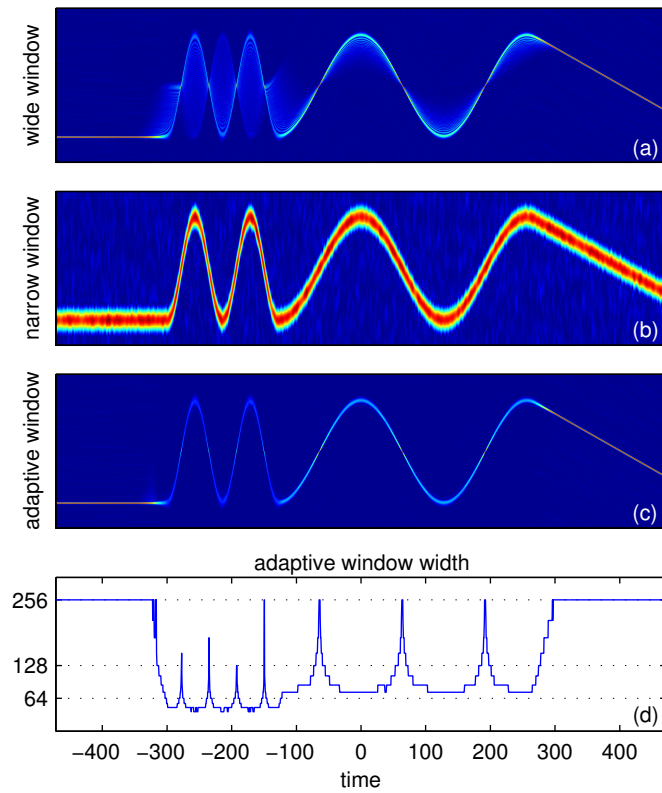


Figure 4: Illustration of the IF estimation using WD with constant and variable windows. FM signal is corrupted by Gaussian white noise with SNR= 20dB. a) WD with wide window. b) WD with a narrow window. c) WD with adaptive window whose width is presented in d).

(a) **The cross WD (XWD)** is proposed to extend the operating SNR range of the WD [30]. It is calculated as

$$\text{XWD}(t, \Omega) = \int_{-\infty}^{\infty} \tilde{x}(t + \tau/2)x^*(t - \tau/2) \exp(-j\Omega\tau) d\tau, \quad (14)$$

where $\tilde{x}(t)$ represents a unit amplitude signal obtained as $\tilde{x}(t) = \exp(j \int_{-\infty}^t \hat{\Omega}(\tau) d\tau)$. The initial IF estimate $\hat{\Omega}(\tau)$ can be obtained by using any estimator, for example, the maxima position of the TF representation (STFT or WD). The IF $\hat{\Omega}(\tau)$ is estimated in an iterative procedure where each step includes the maximization of new $\text{XWD}(t, \Omega)$ that is calculated using the improved unit amplitude signal, obtained with the IF estimate calculated in the previous step. In each iteration we improve the signal energy concentration, increasing the probability of correct IF estimation. The cross PWVD is proposed to improve the IF estimation of polynomial phase signals (PPSs) at lower SNRs [31].

(b) **Modifications of the ICI algorithm.** Based on the estimated SNR and percentage of outliers, set of the window widths in the ICI algorithm is divided into three subsets: 1) subset of the narrowest windows, H_n , that produce poor results for high noise, 2) subset of middle-width windows, H_m , with low percentage of outliers caused by high noise, and 3) subset of wide windows, H_w , that are robust to high noise influence, but usually yield biased estimates. In the modified ICI algorithm, windows from set H_n are not used, since they are characterized by high probability of outliers, whereas those from the $H_m \cup H_w$ subset are used. The IF estimates obtained using windows from H_m have to be filtered (e.g., using median filter) in order to eliminate possible outliers, although the probability of outliers is low. The estimates obtained with H_w windows are used unchanged. The rest of the algorithm is the same as the standard adaptive ICI algorithm described in Section 3. In this way, short windows are eliminated from consideration since they produce significant number of outliers. The algorithm starts with the narrowest window (with the smallest bias) from H_m where the influence of high noise errors is eliminated. Then the algorithm proceeds towards wider windows until the optimal window width is reached.

(c) **Tracking filters in the TF plane.** The IF estimation errors due to high noise are of impulsive nature (see Fig. 5). In this case, the median filter could be used to remove IF estimates characterized by significant deviation from the neighboring IFs. The accuracy of median is limited due to the correlation between errors at consecutive samples. More sophisticated techniques are then required. One such technique considers the Viterbi algorithm [32] for the IF tracking. The algorithm is based on two criteria: (A) IF estimate should pass through the highest values of the TF representation, and (B) the IF variation between

consecutive instants should be as small as possible. The IF estimate can be represented as a path $(n, k(n))$, $n = n_1, n_1 + 1, \dots, n_2$, in the TF plane that minimizes the following function [33]:

$$\sum_{n=n_1}^{n_2} f(\text{TF}(n, k(n))) + \sum_{n=n_1}^{n_2-1} g(|k(n) - k(n+1)|). \quad (15)$$

In (15), the first sum considers criterion (A) ($f(\cdot)$ is a non-increasing function), while the second sum considers criterion (B) ($g(\cdot)$ is a non-decreasing function). For the considered instant n , function f can be formed as follows. First, $\text{TF}(n, k)$ is sorted into nonincreasing order, i.e., frequency indices k_j , $j = 1, 2, \dots, Q$, are determined so that the following relation holds:

$$\text{TF}(n, k_1) \geq \text{TF}(n, k_2) \geq \dots \geq \text{TF}(n, k_Q). \quad (16)$$

The function f is then formed as $f(\text{TF}(n, k_j)) = j - 1$, which corresponds to the idea that larger transform values yield smaller $f(\text{TF}(n, k_j))$ values and therefore are more important candidates for the IF estimates. For example, zero value of $f(\text{TF}(n, k_j))$ corresponds to the maximum value of $\text{TF}(n, k)$. For function $g(|x - y|)$, we can take linear form with respect to $|x - y|$, for example, $g(|x - y|) = c(|x - y| - \Delta)$ for $|x - y| > \Delta$ and zero elsewhere. The threshold is denoted by Δ .

4. Parametric and Combined IF estimators

The most common parametric signal model is the PPS model, or the polynomial FM model, that is grounded by the Weierstrass theorem. The PPS can be defined as $x(t) = A \exp(j \sum_{k=0}^P a_k t^k)$ where P is the polynomial order and $\{a_k | k \in [0, P]\}$ are the phase coefficients. The IF of the PPS equals $\Omega(t) = \sum_{k=1}^P k a_k t^{k-1}$ and it comprises all phase coefficients except a_0 . For $P = 1$ (complex sinusoid), the FT represents an appropriate estimation tool since the signal energy is ideally concentrated at a single point in the frequency domain. For $P > 1$, the signal energy is spread over a frequency range. In that case, it would be convenient to use an operator that can transform the PPS into a complex sinusoid. The maximum likelihood (ML) estimation of the phase parameters requires multidimensional search, which can be computationally exhaustive procedure. The PPS parameters can be efficiently estimated by reducing the polynomial order in the signal phase and estimating the phase parameters, starting from the highest one. The most popular estimators from this class are the high order ambiguity function (HAF) [34], and its variations [35–37].

To transform the signal $x(t)$ to a complex sinusoid, the HAF uses the phase difference (PD) operator

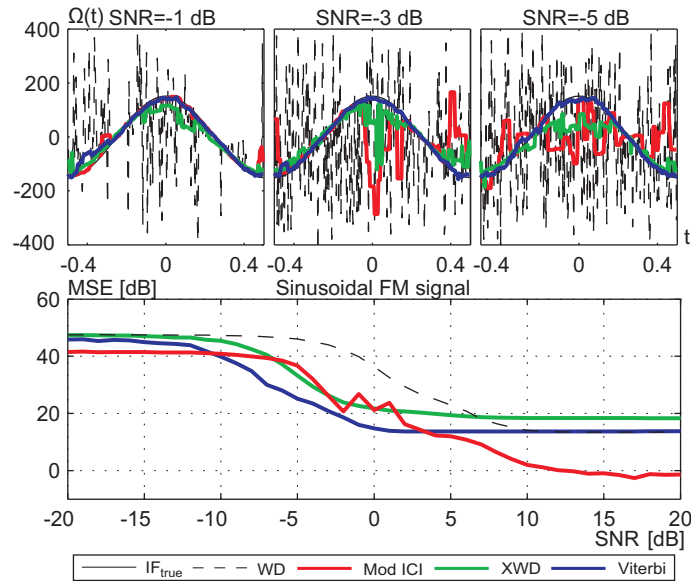


Figure 5: *Top*: IF estimation of a sinusoidal FM signal corrupted by high Gaussian noise using the WD. The position of maximum works inaccurately for all three considered SNRs. The modified ICI, XWD and Viterbi algorithm are close to exact IF for SNR=-1dB. For lower SNR modified ICI has considerable errors, while the XWD can still be used for SNR=-3dB. For the lowest SNR=-5dB only the Viterbi algorithm produces satisfactory results. *Bottom*: MSE for several IF estimators for the above signal. The estimator based on the position of maximum has breakdown point about SNR=8dB and that below this value it becomes extremely inaccurate, XWD estimator has higher bias but it can work for SNR=0dB, the modified ICI algorithm compensates the bias for high SNR and it has better accuracy than the estimator based on the position of maximum while for low SNR it has similar accuracy like XWD algorithm. The Viterbi algorithm seems the most accurate for low SNR what is paid by calculation complexity.

$$\begin{aligned} \text{PD}^p[t; \tau_1^{(l)}, \dots, \tau_p^{(l)}] &= \text{PD}^{p-1}[t + \tau_p^{(l)}; \tau_1^{(l)}, \dots, \tau_{p-1}^{(l)}] \\ &\times \left(\text{PD}^{p-1}[t - \tau_p^{(l)}; \tau_1^{(l)}, \dots, \tau_{p-1}^{(l)}] \right)^* , \end{aligned} \quad (17)$$

where $p = 1, 2, \dots, P - 1$, represents the PD order, and $\tau_i^{(l)}$, $i = 1, \dots, p$, are the lag parameters. In addition, $\text{PD}^0[t] = x(t)$. Since the frequency of sinusoid equals $\Omega = 2^{P-1} P! a_P \prod_{i=1}^{P-1} \tau_i^{(l)}$, the highest-order phase parameter is estimated as follows:

$$\begin{aligned} \hat{a}_P &= \frac{1}{2^{P-1} P! \prod_{i=1}^{P-1} \tau_i^{(l)}} \arg \max_{\Omega} |\text{HAF}^{(l)}(\Omega)|^2, \\ \text{HAF}^{(l)}(\Omega) &= \int_{-\infty}^{\infty} \text{PD}^{P-1}[t, \tau_1^{(l)}, \dots, \tau_{P-1}^{(l)}] \exp(-j\Omega t) dt. \end{aligned}$$

Once a_P is estimated, a_{P-1} can be obtained from the dechirped signal $x(t) \exp(-j\hat{a}_P t^P)$. The procedure is repeated until all the phase parameters are estimated. However, numerous problems appear in the estimation process, namely a) noise influence increases with each PD leading to increased estimation MSE and high SNR threshold, b) appearance of cross-terms when multicomponent signals are considered, and c) error propagation from higher to lower order phase parameters. To reduce the influence of cross-terms, the product HAF (PHAF) is proposed as the product of several HAFs calculated using different sets of lag parameters [35]

$$\text{PHAF}(\Omega) = \prod_l \text{HAF}^{(l)}(F_l \Omega),$$

where $F_l = \prod_{i=1}^{P-1} \tau_i^{(l)} / \tau_i^{(1)}$ is the frequency scaling factor. The scaling in frequency ensures that, in all the HAFs, the auto-terms appear at the same frequency Ω proportional to $\prod_{i=1}^{P-1} \tau_i^{(1)}$. The frequencies of cross-terms, however, are not proportional to this product after scaling in frequency [35]. Multiplying HAFs, therefore, enhances the auto-terms more significantly than cross-terms.

If the presented higher-order representations, including possible post-processing of the IF estimates, cannot produce satisfactory results in high noise case, it is recommended to resort to linear signal representations. The optimal processing can be performed if the signal form is known (matched filters). In many cases, we can assume that the signal phase can be locally approximated by a polynomial (using a finite Taylor series), reducing the analysis to the PPS case. In the TF analysis, this is referred to as the local

polynomial FT (LPFT) [38, 39]

$$\text{LPFT}(t, \mathbf{\Omega}) = \int_{-\infty}^{\infty} x(t + \tau) w(\tau) e^{-j\theta(\mathbf{\Omega}, \tau)} d\tau, \quad (18)$$

where $w(\tau)$ is the sliding localization window, $\theta(\mathbf{\Omega}, \tau) = \Omega\tau + \Omega_2\tau^2/2! + \Omega_3\tau^3/3! + \dots + \Omega_M\tau^M/M!$ and M is the LPFT order. The LPFT is an M -dimensional transform which concentrates at $\mathbf{\Omega} = (\phi^{(1)}(t), \phi^{(2)}(t), \dots, \phi^{(M)}(t))$, and the IF and its derivatives can be estimated as

$$\hat{\mathbf{\Omega}}(t) = \arg \max_{\mathbf{\Omega}} |\text{LPFT}(t, \mathbf{\Omega})|,$$

where the IF is the first element of $\hat{\mathbf{\Omega}}(t)$. Note that the STFT represents a special case of the LPFT obtained with $M = 1$. Due to time localization by the sliding window, the LPFT of relatively low orders could be used, e.g., $M = 2$ or 3 , even if the phase order P is much higher. Also the time localization enables the LPFT to deal with signals that have nonpolynomial modulation. The LPFT is a linear transform, and it requires a search over M parameters for each instant. The search for optimal parameters is a compromise between accuracy and complexity. Direct search over parameter space could be performed when accuracy is crucial.

An alternative approach is the phase unwrapping, where the PPS parameters are estimated directly from the signal phase. It produces accurate estimates only at higher SNRs. However, the phase unwrapping approach can be used to improve the accuracy of methods that can work at lower SNRs [40]. Assuming that the initial estimates of the phase parameters, \hat{a}_k , are obtained by some standard technique, the dechirping of the signal $x(t) \exp(-j \sum_{k=1}^P \hat{a}_k t^k)$ yields a PPS with parameters $\{a_0, \delta a_k = a_k - \hat{a}_k | k \in [1, P]\}$. Then, the phase unwrapping (coupled with signal decimation and filtering) can be used to estimate errors δa_k , thus refining initial estimates. The algorithm is presented in dashed box in Fig. 7.

The IF estimators, robust to noise influence, should be used in the initial stage (prior to refinement). The TF representations, in particular the linear ones, are the main candidates for this purpose. For example, the STFT with various window widths, $\text{STFT}_h(t, \Omega)$, can be used for obtaining the initial IF estimate $\hat{\Omega}_h(t)$. Approximation of $\hat{\Omega}_h(t)$ by a polynomial, i.e., $\hat{\Omega}_h(t) \approx \sum_{k=1}^P k \hat{a}_{k,h} t^{k-1}$, yields rough estimation of the signal parameters $\{\hat{a}_{k,h} | k \in [1, P]\}$, which in turn can be refined using the approach [40], giving estimates $\{\hat{a}_{k,h}^{(f)} | k \in [0, P]\}$. The procedure is repeated for all the considered window widths h , and for the optimal window width, \hat{h} , we declare the one that satisfies $\hat{h} = \arg \max_h |\int_t x(t) \exp(-j \sum_{k=0}^P \hat{a}_{k,h}^{(f)} t^k dt)|$.

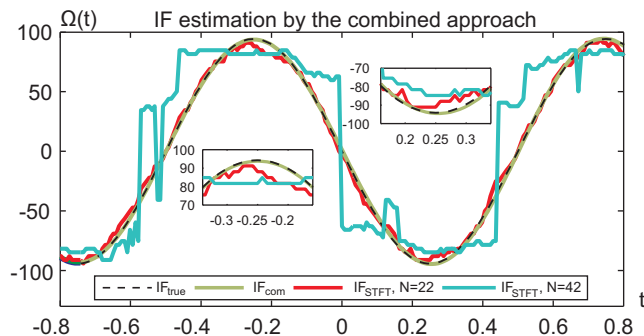


Figure 6: The IF of a sinusoidal FM signal estimated by the QML approach and STFT. SNR=0 dB. The QML estimator produces estimates that are very close to the true IF frequency. The IF estimates obtained by the STFT ($N = 42$) are significantly worse than those obtained by the STFT ($N = 22$).

This technique could be extended to deal with unknown phase order P , as well as with nonpolynomial modulations. The performance of this combined approach, referred to as the quasi-ML (QML) [41], is evaluated on a sinusoidal FM signal. In Fig. 6, the IF estimations obtained by using the STFTs with fixed window widths are compared with the QML approach. The QML estimation algorithm is summarized in the right side in Fig. 7.

5. Conclusions

The IF concept and its connection with the TF representations has been presented. The asymptotic performance measures of the TF-based IF estimators are analyzed and illustrated. The adaptive window width algorithm for the IF estimation is explained. In addition, the IF estimation of signals corrupted by high noise is presented, and strategies to deal with high noise disturbances are summarized. Some of the recent trends in the TF analysis inspired by the IF are reviewed, including a technique for joining nonparametric and parametric estimators.

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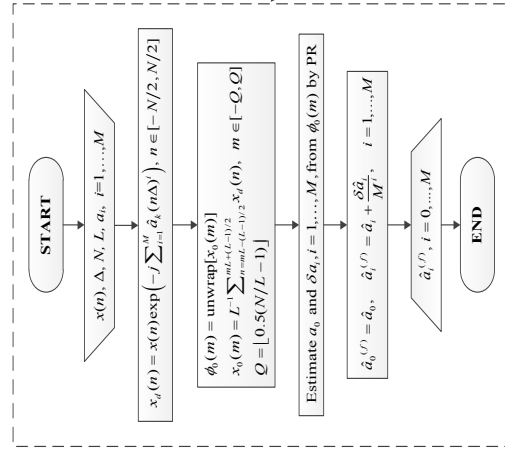
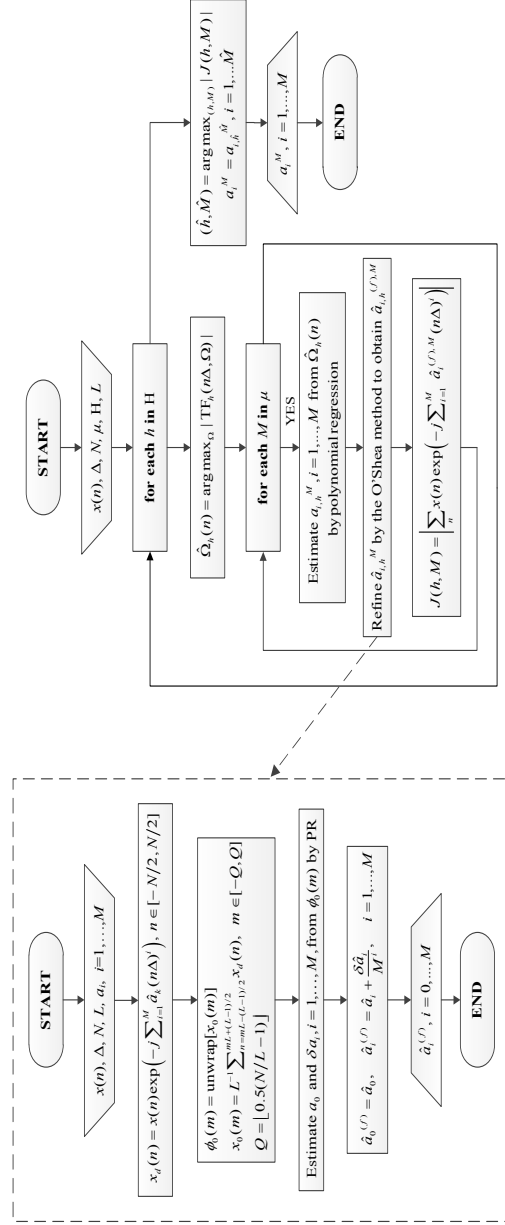


Figure 7: *Left (dashed box)*: Fine estimation procedure based on the phase unwrapping. *Right*: Algorithm for quasi-ML IF estimation.

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