

Nonsparsity Influence on the ISAR Recovery from a Reduced Set of Data

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Abstract—The analysis of ISAR image recovery from a reduced set of data presented in [1] is extended in this correspondence to an important topic of signal nonsparsity (approximative sparsity). In real cases the ISAR images are noisy and only approximately sparse. Formula for the mean square error in the nonsparse ISAR, reconstructed under the sparsity assumption, is derived. The results are tested on examples and compared with statistical data.

Keywords—Radar imaging, ISAR, noisy signal, sparse signal, compressive sensing

I. INTRODUCTION

The inverse synthetic aperture radar (ISAR) is a method for obtaining an image of a target in the range/cross-range domain based on the change in viewing angle with respect to the fixed radar [2]–[4]. Application of compressive sensing methods to the ISAR data is an intensively studied topic, [5]–[9]. A reduced set of data in the ISAR imaging is analyzed in [1] as well. In real cases ISAR images are only approximately sparse. In this correspondence, the analysis of nonsparsity influence to the reconstructed images extends the results from [1]. Simple formula for the mean square error (MSE) in the reconstructed image is derived. The results are tested on examples and compared with statistical data in the cases of nonsparse ISAR images reconstructed under the assumption that they were sparse.

The organization of this correspondence is as follows. After a short review of the results from [1] in Section II, the main result is presented in a form of a theorem in Section III. The result is statistically checked on examples in Section IV.

II. REVIEW

A radar output signal can be modeled as a sum of the signals reflected from individual scattering points. The received signal from the i th scattering point, after an appropriate demodulation, range compensation, and residual video phase filtering, can be defined as

$$q_i(m, n) = \sigma_i e^{j2\pi\beta_i m/M} e^{j2\pi\gamma_i n/N}, \quad (1)$$

where σ_i is the reflection coefficient, β_i and γ_i are the cross-range and range coefficients, respectively (they depend on the radar parameters as well [1]–[4]), M is the number of pulses, and N is the number of samples within each pulse.

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The signal for K scattering points is

$$q(m, n) = \sum_{i=1}^K q_i(m, n). \quad (2)$$

The two-dimensional Fourier transform of the signal $q(m, n)$, assuming that some parts (samples or blocks of samples) of the signal are not available, is estimated as

$$Q(k, l) = \sum_{m=0}^{M-1} \sum_{n \in \mathbb{N}_A(m)} q(m, n) e^{-j(\frac{2\pi mk}{M} + \frac{2\pi nl}{N})}, \quad (3)$$

where $\mathbb{N}_A(m)$ represents the set of available samples within the m -th pulse. The total number of available samples N_A satisfies $1 \ll N_A \leq MN$. The presented model could be applied to the SAR and other imaging systems.

For a large number of randomly positioned unavailable samples $MN - N_A$ the value of $Q(k, l)$ is a sum of terms with quasi arbitrary phases (for k and l not corresponding to β_i and γ_i). It can be considered as a complex-valued variable (missing samples noise) with Gaussian distributed real and imaginary parts, as shown in [10], [13]. Its variance is

$$\text{var}\{Q(k, l)\} = N_A \frac{MN - N_A}{MN - 1} |\sigma_i|^2. \quad (4)$$

For K scattering points we may write [10], [13]

$$\text{E}\{Q(k, l)\} = \sum_{i=1}^K \sigma_i N_A \delta(k - \beta_i, l - \gamma_i) \quad (5)$$

$$\text{var}\{Q(k, l)\} = N_A \frac{MN - N_A}{MN - 1} \sum_{i=1}^K |\sigma_i|^2 (1 - \delta(k - \beta_i, l - \gamma_i)), \quad (6)$$

where $\delta(k, l) = 1$ only for $k = l = 0$ and $\delta(k, l) = 0$, elsewhere.

Assume that an additive noise $\varepsilon(m, n)$ exists in the available data $q(m, n)$. When the recovery is achieved, accuracy of the result is related to the input additive noise [1], [11] only. The energy of noise in the reconstructed signal, assuming only $K \ll MN$ nonzero coefficients (K -sparse signal), is

$$E_{\varepsilon R} = \frac{K}{MN} \frac{M^2 N^2}{N_A^2} \sum_{m=0}^{M-1} \sum_{n \in \mathbb{N}_A(m)} |\varepsilon(m, n)|^2. \quad (7)$$

The SNR in the recovered signal is

$$\begin{aligned} SNR &= 10 \log \frac{E_s}{E_{\varepsilon R}} = 10 \log \frac{\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |q(m, n)|^2}{\frac{K}{N_A} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |\varepsilon(m, n)|^2} \\ &= 10 \log \frac{E_s}{\frac{K}{N_A} E_{\varepsilon}}, \end{aligned} \quad (8)$$

where E_s is the energy of input signal. In the case of white noise the energy of input noise in all samples would be $E_\varepsilon = MN\sigma_\varepsilon^2$, where σ_ε^2 is the input additive noise variance.

III. NONSPARSE SIGNAL RECONSTRUCTION

In general, the signal sparsity assumption in a real data does not hold in a strict sense. Even one point scatterer produces a nonsparse signal if the cross-range and range coefficients β_i and γ_i are not integers (on the discrete-frequency grid). The assumption that the received ISAR signal is sparse is only an approximation¹. General bounds for the reconstruction error for nonsparse signals, reconstructed with the sparsity assumption, are given in [12]. The exact relation for the considered ISAR problem is given by the next theorem.

Theorem: Consider a nonsparse two-dimensional radar signal $q(m, n)$ whose Fourier transform is $Q(k, l)$, with vector notations \mathbf{q} and \mathbf{Q} , respectively. Total number of the signal samples is $M \times N$. Assume that the input additive noise $\varepsilon(m, n)$ is white, with variance σ_ε^2 . Assume that the signal samples at N_A positions, defined by $(m, n) \in \mathbb{N}_A$, are available. The signal is reconstructed under the assumption as if were K -sparse (with the assumption that the reconstruction conditions are met). The reconstructed signal with K nonzero coefficients at $(k, l) \in \mathbb{K}$ is denoted by \mathbf{Q}_R . The error in the reconstructed coefficients with respect to the K corresponding coefficients in the original signal is:

$$\|\mathbf{Q}_K - \mathbf{Q}_R\|_2^2 = K \frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2 + K \frac{(MN)^2}{N_A} \sigma_\varepsilon^2, \quad (9)$$

where \mathbf{Q}_K is equal to the original signal Fourier transform \mathbf{Q} at the reconstructed positions, $Q_K(k, l) = Q(k, l)$ for $(k, l) \in \mathbb{K}$ and $Q_K(k, l) = 0$ for $(k, l) \notin \mathbb{K}$. The assumed nonzero coefficients are above the additive noise level.

Proof: According to (4) and (5) the missing samples in the initial two-dimensional Fourier transform can be represented as a noise. It has been assumed that the assumed sparsity (number of components) K and the measurements matrix satisfy the reconstruction conditions. Then a reconstruction algorithm can detect K signal components whose amplitudes in the time domain are $(\sigma_1, \sigma_2, \dots, \sigma_K)$ and perform signal reconstruction. In the simulations we used the algorithm presented in [1], Section III.A (any of its three presented forms can be used). Additional details of this reconstruction algorithm (including a MATLAB code that can be used to reconstruct the presented results) may be found in [16]. The result of this algorithm, or any other reconstruction algorithm which explicitly uses the fact that the resulting signal is sparse with a sparsity K , is a reconstructed K -sparse signal \mathbf{Q}_R . The remaining

nonreconstructed $MN - K$ signal components with amplitudes $(\sigma_{K+1}, \sigma_{K+2}, \dots, \sigma_{MN})$ produce noise in these K reconstructed components. The variance of noise from a nonreconstructed signal component with amplitude σ_i is defined by (4) as

$$|\sigma_i|^2 N_A (MN - N_A) / (MN - 1). \quad (10)$$

The variance is increased after the reconstruction. The signal amplitudes in $Q(k, l)$, defined by (5), are proportional to N_A . The amplitudes are restored during the reconstruction to their correct values, proportional to MN (as in the case when all signal samples were available). The scaling factor for the reconstructed amplitudes is MN/N_A . Then the scaling factor for the noise variance in the reconstructed components is $(MN/N_A)^2$. It means that in these components the noise variance from a nonreconstructed component is

$$|\sigma_i|^2 \frac{M^2 N^2}{N_A^2} \frac{N_A (MN - N_A)}{MN - 1} \cong |\sigma_i|^2 MN \frac{MN - N_A}{N_A}. \quad (11)$$

The total energy of white noise in K reconstructed components of \mathbf{Q}_R will be K times greater than the variance in one reconstructed component. Total noise caused by the nonreconstructed components $(\sigma_{K+1}, \sigma_{K+2}, \dots, \sigma_{MN})$, is

$$\|\mathbf{Q}_R - \mathbf{Q}_K\|_2^2 = KMN \frac{MN - N_A}{N_A} \sum_{i=K+1}^{MN} |\sigma_i|^2. \quad (12)$$

Energy of the remaining signal, when K components are removed from the original signal (corresponding to the remaining nonreconstructed components), will be denoted by

$$\|\mathbf{Q} - \mathbf{Q}_K\|_2^2 = \sum_{i=K+1}^{MN} |MN\sigma_i|^2. \quad (13)$$

From (12) and (13) we get

$$\|\mathbf{Q}_R - \mathbf{Q}_K\|_2^2 = K \frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2. \quad (14)$$

If the original signal is K -sparse, i.e. $\mathbf{Q} = \mathbf{Q}_K$, then there is no error

$$\|\mathbf{Q}_R - \mathbf{Q}_K\|_2^2 = 0. \quad (15)$$

The same result $\|\mathbf{Q}_R - \mathbf{Q}_K\|_2^2 = 0$ follows if all signal samples are available, $MN = N_A$. Obviously, if a complete set of samples is used, then the error is zero for any sparsity.

Consider now that a nonsparse signal $q(m, n)$ has an additive complex-valued noise. According to the results in [1], [13], [14], with the assumption that all reconstructed amplitudes are above the additive noise level, this noise can be considered as additive after reconstruction as well. The total error in the reconstructed signal, with respect to the original signal at the same coefficient positions, is

$$E_{\varepsilon R} = \frac{K}{N_A} E_\varepsilon = MN \frac{K}{N_A} \sigma_\varepsilon^2. \quad (16)$$

The noise energy in the Fourier domain is multiplied by a factor MN as well, since $\|\mathbf{Q}\|_2^2 = MN \|\mathbf{q}\|_2^2$. It means that for a noise only case we would get

$$\|\mathbf{Q}_K - \mathbf{Q}_R\|_2^2 = MNE_{\varepsilon R} = (MN)^2 \frac{K}{N_A} \sigma_\varepsilon^2. \quad (17)$$

¹The sparsity degradation in the off-grid cases (basis mismatch) problem can be reduced by signal oversampling, making the frequency grid finer (increasing the number of basis functions), at the expense of the computation complexity. Efficiency of a reconstruction algorithm in the ISAR can be improved taking into account the property that scattering points are usually grouped in space and by recovering the signal under the group sparsity constraint.

Including both components (14) and (17) we get the theorem result

$$K \frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2 + K \frac{(MN)^2}{N_A} \sigma_\varepsilon^2. \quad (18)$$

This completes the proof.

The relation (9) can be written in a form of the reconstructed error, normalized with the number of reconstructed components, as

$$\frac{1}{K} \|\mathbf{Q}_K - \mathbf{Q}_R\|_2^2 = \frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2 + \frac{(MN)^2}{N_A} \sigma_\varepsilon^2. \quad (19)$$

IV. EXAMPLES

Example 1: Consider a nonsparse signal

$$q(m, n) = \frac{1}{MN} \left(\sum_{i=1}^{MN} \sigma_i e^{j2\pi(\beta_i m/M + \gamma_i n/N)} + \varepsilon(n) \right), \quad (20)$$

where $0 \leq \beta_i < M$ and $0 \leq \gamma_i < N$ are random frequency indices. Signal amplitudes are normalized $\sigma_i = 1$ for $i = 1, \dots, S$, while the components for indices above S are $\sigma_i = e^{-2i/(S+1)}$, $i = S + 1, S + 2, \dots, MN$. The signal can be considered as an approximately S -sparse signal. Using $M = N = 64$ and $S = 50$, the first K components of the signal are reconstructed for various assumed sparsity within a wide range $K = 24, 26, 28, \dots, 48, 50, 52, \dots, 98, 100$, which are below and above sparsity S . The remaining $MN - K$ signal components behave as disturbance. Additive noise is Gaussian with standard deviation of the real and imaginary part $\sigma_\varepsilon = 0.1$. Reconstruction of the nonsparse signal $q(m, n)$ for the assumed values of sparsity K is done. The average squared error in 20 realizations with random frequency value positions and positions of the available N_A samples is calculated. The results with $N_A = 1024 = MN/4$ and $N_A = 2048 = MN/2$ available samples (25% and 50% of the total number of samples) are presented. The normalized error energies in the frequency domain (normalized to the assumed sparsity) are calculated according to the theorem and (19)

$$E_{statistics} = 10 \log \left(\frac{1}{K} \|\mathbf{Q}_K - \mathbf{Q}_R\|_2^2 \right) \quad (21)$$

$$E_{theory} = 10 \log \left(\frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2 + \frac{(MN)^2}{N_A} \sigma_\varepsilon^2 \right). \quad (22)$$

They are given in Fig.1 and Fig.2, as a function of the assumed sparsity K . The theoretical values are plotted with solid line. The statistical data are presented by dots with average values (presented by circles) which almost coincides with the theory in both cases. The simulation is repeated with 50% of the total number of samples and a stronger noise whose standard deviation is $\sigma_\varepsilon = 0.5$, Fig.3.

Example 2: The same analysis is done on the ISAR data according to the delta-wing experiment described in [15]. The

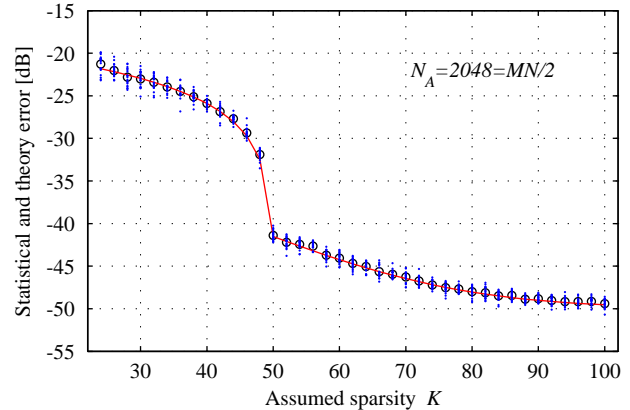


Fig. 1. Error energy in the reconstruction of nonsparse signal calculated statistically (dots for realizations and circles for average) and according to the theory (solid line) for various assumed sparsity and 50% of the available samples.

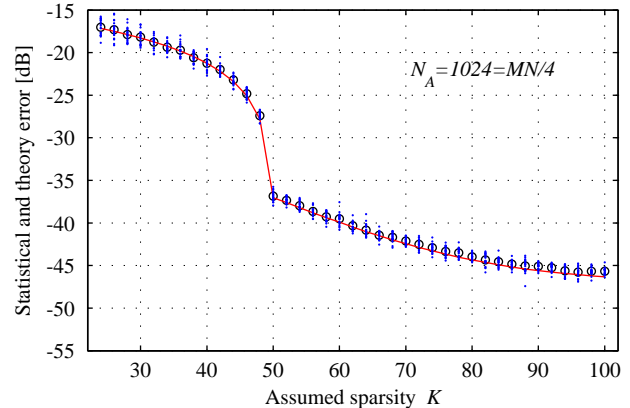


Fig. 2. Error energy in the reconstruction of nonsparse signal calculated statistically (dots for realizations and circles for average) and according to the theory (solid line) for various assumed sparsity and 25% of the available samples.

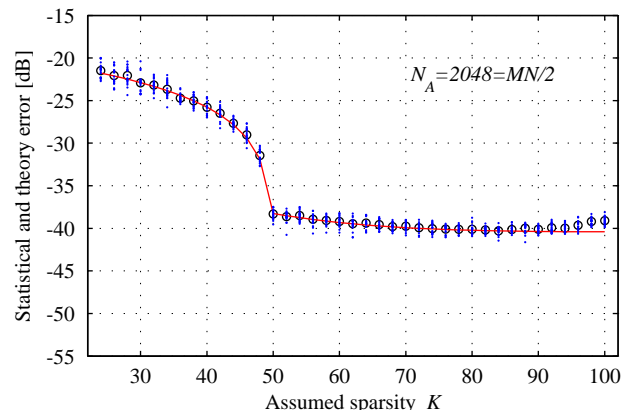


Fig. 3. Error energy in the reconstruction of nonsparse signal calculated statistically (dots for realizations and circles for average) and according to the theory (solid line) for various assumed sparsity and 50% of the available samples with $\sigma_\varepsilon = 0.5$.

TABLE I

ERROR ENERGY IN THE ISAR RECONSTRUCTED COEFFICIENTS WITH 33% AVAILABLE SAMPLES FOR VARIOUS ASSUMED SPARSITY K .

$N_A = MN/3$	$K = 6$	$K = 9$	$K = 12$	$K = 48$
<i>Theory</i>	-24.85	-26.42	-27.67	-34.39
<i>Statistics</i>	-24.66	-26.39	-27.88	-34.01

experiment was conducted by using an X-band radar operating at a center frequency of 10.1 GHz with 300 MHz bandwidth and a range resolution of 0.5 m. The pulse repetition time is $T_r = 1/2000 = 0.5$ ms. The total data set used in this example contains samples for 2048 range profiles with 50 bins. The target was a delta-wing shaped apparatus. It consisted of six-scatterer model. The target model has a length of 5 m on each of its three sides of regular triangle. The delta-wing is at a range of 2 km and was rotating at 3 degree/s. Data within the interval of 50 range bins (where the target was located) are shown only. The original ISAR image with all available samples is presented in Fig.4(a), (b). Assuming different sparsities the results with a third of the available samples are presented in Fig.4. The accuracy of the reconstructed signal components (with respect to original ones with all signal samples used in the calculation) is proportional to the energy of the remaining signal content (above the assumed sparsity). The square error, normalized to the assumed sparsity K and the maximal coefficient absolute value, is calculated using

$$E_{statistics} = 10 \log \left(\frac{\frac{1}{K} \|\mathbf{Q}_K - \mathbf{Q}_R\|_2^2}{\max_{k,l} \{|Q(k,l)|^2\}} \right) \quad (23)$$

$$E_{theory} = 10 \log \left(\frac{\frac{MN - N_A}{N_A MN} \|\mathbf{Q} - \mathbf{Q}_K\|_2^2}{\max_{k,l} \{|Q(k,l)|^2\}} \right). \quad (24)$$

The errors are checked statistically by using 33% of the total number of samples as available samples, $N_A = MN/3$, at random positions in 100 realizations. These errors in K reconstructed components, obtained statistically and by using the energy of remaining components, are given in Table I.

V. CONCLUSIONS

In this correspondence, we examined the influence of non-sparsity to the ISAR image reconstruction, using the method of the sparse signal processing. From the main result we can conclude that the influence of noise increases with an increase of the assumed sparsity in the reconstruction. Since the reconstruction error is also proportional to the energy of the remaining part of the signal, the optimal assumed sparsity will be the smallest one when all signal components above noise level are included. The results are statistically checked. The statistical values confirms theoretical results.

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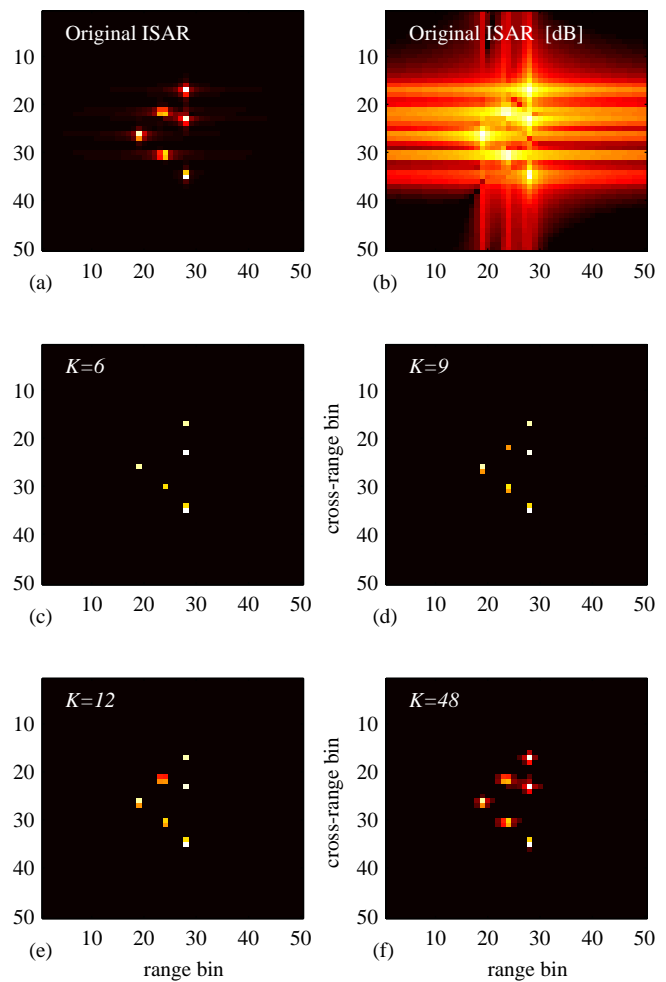


Fig. 4. (a) Original ISAR using all signal samples. (b) The logarithmic scale ISAR with all samples. (c)-(f) Reconstructed ISAR images using $N_A = MN/3$ (33%) available samples and assuming various sparsities, $K = 6$, $K = 9$, $K = 12$, and $K = 48$.

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