Sample Selection Strategy in DFT based Compressive Sensing

Miloš Daković, *Member*, *IEEE*, Isidora Stanković, *Student Member*, *IEEE*,
Joachim Ender, *Fellow*, *IEEE* Ljubiša Stanković, *Fellow*, *IEEE*

***Abstract* — A recently proposed strategy of selecting samples for a unique reconstruction of a signal is analyzed in this paper. The considered signal is sparse in the discrete Fourier transform (DFT) domain. Since the problem is of theoretical importance, we use the basic direct search method for the reconstruction and comparisons of sampling strategies. It is shown that, by using the proposed sampling strategy method, the sparsity limit for the unique reconstruction is increased in comparison to the random selection of samples.**

***Keywords* — compressive sensing, signal reconstruction, sparse signal processing, sample selection**

I. Introduction

A sparse signal is a signal having only few nonzero entries in one of its representation domains. These signals can be reconstructed with a reduced set of measurements/samples. Advantages of compressive sensing in signal transmission and storage are very important, especially in big data setups. The field dealing with reconstruction of sparse signals is known as compressive sensing. It is a growing field in recent years [1]–[11]. Since the introduction of compressive sensing, many reconstruction theorems and algorithms were developed [1]–[7]. In real applications many signals are sparse in a certain domain, representing a ground for wide usage of the compressive sensing theory in different areas of signal processing.

One of the challenging topics in the compressive sensing is the optimal sampling strategy that will allow to reconstruct the signal with smallest possible number of available samples [12]. Various approaches are used to this aim, like those that minimize the coherence index of the isometry constant for a given signal transform. The aim of this paper to find the theoretical minimum of the number of signal samples for the reconstruction of signals sparse in the discrete Fourier transform (DFT) domain. Since various, computationally efficient, reconstruction algorithms require more strict recovery conditions than the uniqueness of the solution requires, here we will use the direct combinatorial search. This method will provide exact reconstruction results if the unique reconstruction is theoretically possible. However, this method is computationally complex. For a large dimension of the reconstruction problem, number of available samples and sparsities it is NP-hard and therefore not computationally feasible. The direct search can be used with small dimensions of problem. Thus, we will test our sampling strategy on the dimensions when the direct combinatorial search is possible. In this way we will avoid specific reconstruction requirements imposed by other reconstruction methods.

The paper is organized as follows. In Section II some basic definitions of compressive sensing are introduced. In Section III the reconstruction algorithm is presented. The procedure for sample selection, using the uniqueness theorem, is presented in Section IV. Results and comparison are shown in Section VI.

II. Basic Definitions

Consider a complex-valued discrete signal $x(n)$ of length $1 \leq n \leq N$. Its discrete Fourier transform (DFT) is denoted by $X(k)$. The signal and its transform are defined as

$$x(n) = \sum_{k=0}^{N-1} X(k)\psi_k(n), \qquad X(k) = \sum_{n=0}^{N-1} x(n)\varphi_n(k).$$

In the vector form, these relations can be written as $\mathbf{x} = \mathbf{\Psi}\mathbf{X}$ and $\mathbf{X} = \mathbf{\Phi}\mathbf{x}$, where $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are the direct and inverse DFT transformation matrices. We assume that the signal is K-sparse in the DFT domain, where $K \ll N$. In this case, we can only use $M < N$ samples to reconstruct the signal of length N. The signal with M samples/measurements/observations will be defined as $y(m)$. Its vector form is $\mathbf{y}$.

The goal of compressive sensing is to minimize the sparsity of $\mathbf{X}$ by knowing a reduced set of the available samples/measurements $\mathbf{y}$. The reconstruction of a signal from a reduced set of data can be formulated as a optimization problem

$$\min \ \|\mathbf{X}\|_0 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{X} \qquad (1)$$

where $\mathbf{y}$ are the samples/measurements and $\mathbf{A}$ is an $M \times N$ measurement matrix. Its rows correspond to the positions of the available samples. The ℓ_0-norm conts the number of nonzero coefficients in $\mathbf{X}$. However, this norm is not convex and its minimization could be done only through a combinatorial search. The direct combinatorial search is an NP-hard problem. This norm is also very sensitive (not applicable) to the noisy signal cases. This the reason why, in practice and theory, more robust norms are used to measure the sparsity. The ℓ_1-norm is the most frequent used norm since it is closest convex function to the ℓ_0-norm. It is equal to the sum of absolute values of $\mathbf{X}$. However, all norm-one reconstruction methods require more samples/measurements than the minimal possible number that can provide unique signal reconstruction in theory. This is the reason why here we will use direct combinatorial approach to the sparse signal reconstruction,

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M. Daković, I. Stanković, Lj. Stanković are with the Faculty of Electrical Engineering, University of Montenegro, Dzordza Vasingtona bb, 81000 Podgorica, Montenegro, e-mail: {milos,isidoras,ljubisa}@ac.me

J. Ender is with Universität Siegen, Zentrum für Sensorsysteme ZESS, Paul-Bonatz-Straße 9-11, 57076 Siegen, Germany e-mail: joachim.ender@uni-siegen.de

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with dimensions when it is computationally feasible (for example, $N \leq 32$).

III. Direct search reconstruction

Any problem described with (1) can be solved by a direct search over whole set of possible values of nonzero coefficient positions. The considered algorithm is the direct search minimisation of the ℓ_0-norm. Assume a vector $\mathbf{X}$ whose sparsity is K. In the direct search we try with all possible combinations of nonzero index values

$$k \in \{k_1, k_2, ..., k_K\} = \mathbf{K}$$

out of the set of all possible indices from 1 to N, i.e. $\{k_1, k_2, ..., k_K\} \subset \mathbf{N}$, where $\mathbf{N} = \{1, 2, ..., N\}$. The vector $\mathbf{X}_K$ contains assumed K nonzero elements of $\mathbf{X}$ at the positions $\mathbf{K}$. The system

$$\mathbf{y} = \mathbf{A}_K\mathbf{X}_K$$

with $M > K$ equations is solved by minimising the least square error

$$e^2 = (\mathbf{y} - \mathbf{A}_K\mathbf{X}_K)^H(\mathbf{y} - \mathbf{A}_K\mathbf{X}_K) = \\ ||\mathbf{y}||_2^2 - 2\mathbf{X}_K^H\mathbf{A}_K^H\mathbf{y} + \mathbf{X}_K^H\mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K. \tag{2}$$

The minimum of the error is found from

$$\frac{\partial e^2}{\partial \mathbf{X}_K^H} = -2\mathbf{A}_K^H\mathbf{y} + 2\mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K = 0. \tag{3}$$

The solution is then

$$\mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K = \mathbf{A}_K^H\mathbf{y} \\ \mathbf{X}_K = \left(\mathbf{A}_K^H\mathbf{A}_K\right)^{-1}\mathbf{A}_K^H\mathbf{y}. \tag{4}$$

To find all possible combinations of $\{k_1, k_2, ..., k_K\} \subset \mathbf{N}$, the total number of systems that should be solved is $\binom{N}{K}$. If, for example, we have $N = 16$ and $K = 3$ then the number of possible combinations is $\binom{16}{3} = \frac{16\times15\times14}{3\times2\times1} = 560$ for $k \in \mathbf{K}$. For all solutions we check the error $\mathbf{y} - \mathbf{A}_K\mathbf{X}_K$. If the mean square error is zero we consider this as the result of the reconstruction for signal $\mathbf{X}$. If there is more than one result, then the reconstruction is not unique.

IV. Reconstruction uniqueness

In the theory of compressive sensing, the restricted isometry property is the condition which has to be satisfied for the signal to be uniquely reconstructed. However, this condition is of high computational complexity and very strict. Also, in many practical cases, it is very pessimistic and discards many settings of the measurement matrix that can produce a unique solution. A uniqueness theorem for signals sparse in the DFT domain, reconstructed using a reduced set of samples, is recently introduced in [13], [14].

The signal sparsity in the DFT domain is K. The DFT is calculated with $N = 2^r$ samples. The solution uniqueness can be checked by using Theorem 1 and Corollary 2. Sparsity limit obtained by Theorem 1 is strict. It has been shown that many cases included in Theorem 1 are zero probability events, leading to a relaxed uniqueness check, formulated by Corollary 2. Zero probability events that reconstructed signal components and their amplitudes are fully correlated to the positions of missing samples are neglected here.

Theorem 1 *Consider a signal $x(n)$ that is sparse in the DFT domain with unknown sparsity. Assume that the signal length is $N = 2^r$ samples and that Q samples are missing at the instants $q_m \in \mathbb{N}_Q$. Assume that the reconstruction is performed and that the DFT of reconstructed signal is of sparsity K. The reconstruction result is unique if the inequality*

$$K < N - \max_{h=0,1,\ldots,r-1}\left\{2^h\left(Q_{2^h} - 1\right)\right\} - K$$

holds. Integers Q_{2^h} are calculated as

$$Q_{2^h} = \max_{b=0,1,\ldots,2^h-1}\left\{\mathrm{card}\{q : q \in \mathbb{N}_Q,\ \mathrm{mod}(q, 2^h) = b\}\right\}$$

Corollary 2 *Consider the signal $x(n)$ that is sparse in the DFT domain. Assume that signal length is $N = 2^r$ samples and that Q samples are missing at the instants $q_m \in \mathbb{N}_Q$. Also assume that the reconstruction is performed and that the DFT of reconstructed signal is of sparsity K. Assume that the amplitudes of signal components are arbitrary with arbitrary phases so that the case when all of them can be related to the values defined by using the missing sample positions is a zero-probability event. The reconstruction result is not unique if the inequality*

$$K \geq N - \max_{h=0,1,\ldots,r-1}\left\{2^h\left(Q_{2^h} - 1\right)\right\} - 1$$

holds. Integers Q_{2^h} are calculated in the same way as in the Theorem 1.

Proof of the Theorem 1 and Corollary 2 is given in [13].

Example: Consider a signal of length $N = 2^4 = 16$ and $M = 10$ available samples which mean that there are $Q = N - M = 6$ missing samples. The goal is to find the maximum sparsity when the uniqueness is satisfied. Also, we will find the limit when we are sure that the signal cannot uniquely be reconstructed. Consider an example of missing samples at the positions

$$q_m \in \mathbb{N}_Q = \{0, 1, 5, 8, 10, 11\}.$$

1) Firstly we take $h = 0$ which means that $Q_{2^0} = Q$ and $2^0(Q_{2^0} - 1) = Q - 1 = 5$.

2) When $h = 1$, the value Q_{2^1} is taken as the maximum value between the total numbers of even and odd positions of the missing samples. That is,

$$\mathrm{card}\{q : q \in \mathbb{N}_Q,\ \mathrm{mod}(q, 2) = 0\} = \mathrm{card}\{0, 8, 10\} = 3. \\ \mathrm{card}\{q : q \in \mathbb{N}_Q,\ \mathrm{mod}(q, 2) = 1\} = \mathrm{card}\{1, 5, 11\} = 3.$$

So $Q_{2^1} = \max\{3, 3\} = 3$ and $2^1(Q_{2^1} - 1) = 4$.

3) For $h = 2$, the total number of missing samples whose positions are a multiple of 4, with starting counting positions $b = 0, 1, 2, 3$, are taken to get the value $Q_{2^2} = \max\{\mathrm{card}\{0, 8\}, \mathrm{card}\{1, 5\}, \mathrm{card}\{10\}, \mathrm{card}\{11\}\} = \max\{2, 2, 1, 1\} = 2$ with $2^2(Q_{2^2} - 1) = 4$.

4) The last case for this N is when $h = 3$. Then Q_{2^3} is found as the maximal number of missing samples with positions at a multiple of 8, with starting counting positions $b = 0, 1, 2, 3, 4, 5, 6, 7$. We calculated that $Q_{2^3} = \max\{2, 0, 1, 1, 0, 0, 0, 0\} = 2$. and $2^3(Q_{2^3} - 1) = 8$.

Going back to the theorem, the signal is considered as uniquely reconstructible, if the sparsity K is

$$K < N - \max_{h=0,1,2,3} \left\{ 2^h \left(Q_{2^h} - 1\right) \right\} - K$$
$$K < 16 - \max\{5, 4, 4, 8\} - K$$
$$K < 4.$$

Using Corollary 2, we can claim that the signal will not be uniquely reconstructed when

$$K \geq 16 - \max\{5, 4, 4, 8\} - 1$$
$$K \geq 7.$$

V. Sampling Strategy

The positions of the samples which are taken for the reconstruction (or missing samples) play a very important role for the uniqueness of the reconstruction. Note that higher values of Q_{2^h} produce lower sparsity limits and vice versa. Also note that the reconstruction will be unique, with a high probability, if the conditions from Corollary 2 are not satisfied.

The simplest case, when $K = 1$, can analytically be analyzed. Consider that we have only $M = 2$ available samples, which is, in this case, enough for the reconstruction. If we select the first sample at any position, the second sample should not be at the even distance from the first sample. The probability of the unique reconstruction failure is then $(N/2 - 1)/(N - 1)$. If we take $M = 3$ available samples, when we set the first sample at an arbitrary position, the second and the third sample should not be at even distance positions from the first one. The probability of unique reconstruction failure is $(N/2-1)(N/2-2)/[(N-1)(N-2)]$ and so on. For M observations the unique reconstruction failure probability is

$$P_F = \frac{(N/2-1)(N/2-2)...(N/2-M+1)}{(N-1)(N-2)...(N-M+1)}. \quad (5)$$

This probability is calculated when the random selection of samples is used.

The sample selection procedure inspired by Theorem 1 was introduced in [14]. The main goal for selection of the samples for the reconstruction is to spread the elements of the set of missing samples as much as possible over each partition and obtain minimal possible Q_{2^h} in Theorem 1 (and Corollary 2). The MATLAB code for proposed selection procedure is given in Algorithm 1.

It means that we have to minimize factors Q_{2^h} for any h. That would be achieved if we equally spread the missing sample positions over sets $\{q : q \in \mathbb{N}_Q,\ \mathrm{mod}(q, 2^h) = b\}$, $b = 0, 1, \ldots, 2^h - 1$. If we write missing sample positions in binary format it would mean that for $h = 1$ an equal number of missing sample positions should have 0 and 1 as the last digit. Then, for $h = 2$ there should be the same number of positions with last two digits 00, 01, 10 and 11, and so on.

The sampling strategy is tested on more examples in the next section.

VI. Results

Consider a signal of length $N = 16$, with sparsity $K = 4$ and $M = 8$ available samples. The reconstruction using the random selection of available samples and selection using the proposed strategy and the uniqueness theorem is shown in the Fig. 1. The reconstruction was done in 100 realizations. It is shown that 99 out of 100 are uniquely reconstructed using the random selection and 100 out of 100 using the uniqueness theorem.

Consider now a more strict case, when $N = 16$ total samples, sparsity $K = 5$ and only $M = 6$ available samples. The reconstruction using random selection of available samples and selection using the uniqueness theorem is shown in the Fig. 2. The reconstruction was done in 100 realizations. In this case, it is shown that only 75 out of 100 are uniquely reconstructed using the random selection and 100 out of 100 using the uniqueness theorem.

Algorithm 1 Samples selection – MATLAB code

```
function Nq = Select_Samples(N,Q)
A = zeros(1,Q);
b = nextpow2(Q);
for k = 0:(b-1)
    for p = 0:(2^k-1)
        S = find(A==p);
        m = round( length(S)/2 + 0.1*(rand-0.5) );
        A(S(1:m)) = A(S(1:m)) + 2^k;
    end
end
B = randperm(N,Q) -1;
Nq = sort( A + 2^b*floor(B/2^b) );
```

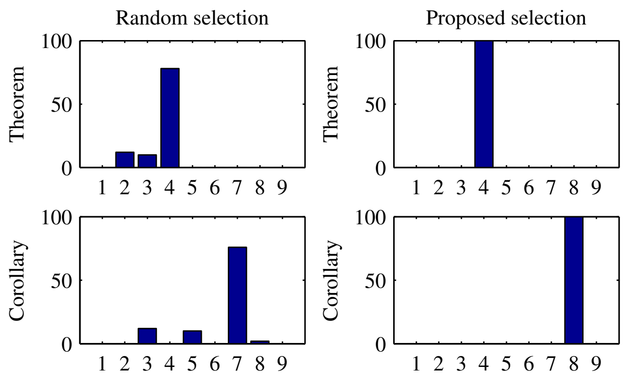

Fig. 1. Sparsity limits in 100 realizations when $N = 16, M = 8, K = 4$ by using random selection (left) and proposed selection (right)

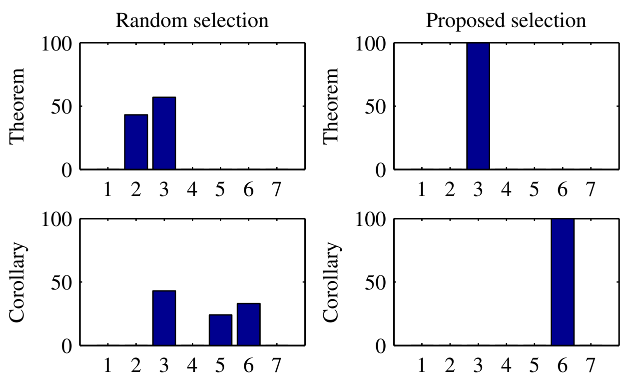

Fig. 2. Sparsity limits in 100 realizations when $N = 16, M = 6, K = 5$ by using random selection (left) and proposed selection (right)

TABLE I
PERCENTAGE OF THE UNIQUELY RECONSTRUCTED SIGNALS WITH $N = 16$ FOR VARIOUS M AND K

	Random selection sparsity K								Deterministic selection sparsity K							
M	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
2	61	×	×	×	×	×	×	×	100	×	×	×	×	×	×	×
3	74	58	×	×	×	×	×	×	100	100	×	×	×	×	×	×
4	93	83	61	×	×	×	×	×	100	100	100	×	×	×	×	×
5	98	97	82	73	×	×	×	×	100	100	100	100	×	×	×	×
6	99	100	94	88	74	×	×	×	100	100	100	100	100	×	×	×
7	100	100	99	90	86	69	×	×	100	100	100	100	100	100	×	×
8	100	100	100	98	94	87	68	×	100	100	100	100	100	100	100	×
9	100	100	99	100	99	94	86	68	100	100	100	100	100	100	100	100
10	100	100	100	99	98	98	97	90	100	100	100	100	100	100	100	100
11	100	100	100	100	100	99	99	97	100	100	100	100	100	100	100	100

TABLE II
PERCENTAGE OF THE UNIQUELY RECONSTRUCTED SIGNALS WITH $N = 32$ FOR VARIOUS M AND K

	Random selection K				Deterministic sel. K			
M	1	2	3	4	1	2	3	4
2	54	×	×	×	100	×	×	×
3	72	78	×	×	100	100	×	×
4	89	92	80	×	100	100	100	×
5	96	91	90	81	100	100	100	100
6	99	95	99	91	100	100	100	100
7	99	100	97	98	100	100	100	100
8	100	99	99	97	100	100	100	100
9	100	100	100	97	100	100	100	100
10	100	100	100	98	100	100	100	100
11	100	100	100	100	100	100	100	100

In Table I, we present the percentage of uniquely reconstructed signals in 100 realizations when $N = 16$ is used. We assume that the number of available samples and the sparsity varies. The sign '$\times$' means that the reconstruction is not possible with these values since $K \geq M$. Note that when number of available samples M is high there is no significant difference between random and deterministic sample selection procedure. The advantage of the deterministic selection is evident when number of available samples M is low.

In Table II are the values for deterministic and random sample selection with $N = 32$ and different M and K. The tables show that in more cases the deterministic way of selection gives better results in comparison to the random selection. Only when we have more available samples, then the results are similar between the two strategies.

VII. CONCLUSION

In this paper we analyzed the optimal strategy for selecting available samples using the uniqueness theorem. The analyzed strategy increases the sparsity limit for an unique reconstruction of a signal. It is statistically shown that the sparsity can be increased, in comparison to the random selection of samples.

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