Combination of Gradient Based and Single Iteration Reconstruction Algorithms for Sparse Signals

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***Abstract*—A new algorithm implemented as combination of gradient based and single iteration reconstruction algorithms for compressively sensed sparse signals is proposed in the paper. A good feature of the gradient algorithm to perform reconstruction for a wide range of applications is combined with the speed of single iteration algorithm in order to perform faster reconstruction in the cases where single iteration algorithm cannot performs reconstruction. The proposed method is of special importance for any application where it is not possible to separate signal components from noise in sparse domain.**

***Keywords*—Gradient algorithm, compressive sensing, sparse signals, signal reconstruction**

I. Introduction

Signals having small number on non-zero coefficients compared to the signal length are called sparse. This kind of signals are present in many practical application. This is the reason why processing of sparse signals attracts significant research interest in the last decade [1]–[4]. The area that exploit the signal sparsity is called compressive sensing (CS). The idea behind compressive sensing is to use just a few measurements rather than all signal samples obtained respecting Nyquist-Shannon sampling theorem. These measurements are linear combinations of sparse domain coefficients [5]. There are two directions in CS. The first one is how to obtain measurements, and the second one is how to reconstruct original signal from these measurements. There are many algorithms which deal with signal reconstruction [6]–[11].

This paper propose a new method combining two algorithms belonging to completely different classes of algorithms. The fist one is called single iteration reconstruction algorithm [10]. It reconstructs signal in sparse domain by separating signal components from noise (caused by missing samples). When positions of signal components are detected, the algorithm reconstruct complete signal in one step. However, depending on number of missing samples (amount on noise), detection of signal components is not always possible. The second algorithm used in proposed method is called gradient based signal reconstruction algorithm [11]. It is iterative algorithm, reconstructing time-domain missing samples in order to produce minimal concentration measure in sparse domain. It was shown that this algorithm can be used for wide range of applications [12], [13]. The idea behind proposed method is to use gradient algorithm in order to reduce noise in sparse domain, and then to apply effective single iteration algorithm to reconstruct whole signal, when signal components can be detected.

Without loss of generality, this paper will consider Discrete Fourier Domain as the sparsity domain of the signal.

II. Background

Consider N samples of time-domain signal $x(n)$, $n = 0,1,\ldots,N-1$. The DFT coefficients of this signal will be denoted with $X(k)$, where $k = 0,1,\ldots,N-1$. These two signals are related via:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \tag{1}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}. \tag{2}$$

In the matrix notation, above equations can be written as: $\mathbf{X} = \mathbf{W}\mathbf{x}$ and $\mathbf{x} = \mathbf{W}^{-1}\mathbf{X}$, where $\mathbf{X}$ is vector having elements $X(k)$, and $\mathbf{x}$ is vector of $x(n)$ elements. Both vectors are of length N, and $\mathbf{W}_{N\times N}$ is transform matrix with elements $\exp(-j2\pi nk/N)$.

In the case when most of coefficients of signal $X(k)$ are zero-valued or negligible, we may say that $X(k)$ is sparse presentation of signal $x(n)$. Compressive sensing reconstruction algorithms are used to reconstruct sparse signals from reduced set of measurements. Measurements are considered to be linear combinations of sparse domain coefficients.

Consider M randomly positioned samples of signal $x(n)$ denoted by $y(i) = x(n_i)$, $i = 1,2,\ldots,M$ (y in vector notation). The set of these samples positions will be denoted by $\mathbb{N}_A \in \{n_1, n_2, \ldots, n_M\}$, while its component containing the positions of all other samples will be denoted with $\mathbb{N}_Q$. These samples may be obtained as a linear combination of $X(k)$ coefficients $\mathbf{y} = \mathbf{A}\mathbf{X}$. In this case, matrix $\mathbf{A}_{M\times N}$ is obtained from matrix $\mathbf{W}^{-1}$ by preserving M rows at positions $\mathbb{N}_A$, while the rows

This research is supported by the Montenegrin Ministry of Science, project grant CS-ICT "New ICT Compressive sensing based trends applied to: multimedia, biomedicine and communications".

978-1-5090-3843-5/17/$31.00

at position $\mathbb{N}_Q$ are removed. The compressive sensing task is to reconstruct signal $X(k)$ (or $x(n)$) from $\mathbf{y}$. Signal samples at positions $\mathbb{N}_A$ will be called available, while the ones from $\mathbb{N}_Q$ will be called missing or omitted. Note that missing samples produce noise in sparse domain, and even for one missing samples, all zero valued coefficients in sparse domain will assume some nonzero value.

A. Single Iteration Reconstruction Algorithm - SIRA

Single iteration reconstruction algorithm [10] is based on the idea of separating signal components from noise in sparse domain. In the case when all signal components are detected, the signal can be reconstructed as:

$$\mathbf{X}_R = \left(\mathbf{A}_{CS}^T \mathbf{A}_{CS}\right)^{-1} \mathbf{A}_{CS}^T \mathbf{y}. \tag{3}$$

Matrix $\mathbf{A_{CS}}$ is obtained from $\mathbf{W^{-1}}$ by selecting rows at positions $\mathbb{N}_A$, and columns at the positions corresponding to positions of signal coefficients in sparse domain of the signal. Under assumption that all signal components are detected, the reconstruction can be performed very fast. However, by increasing number of missing samples, the noise in sparse domain can surpass signal coefficients, and then it is impossible to detect signal components and therefore, reconstitution can not be performed.

B. Gradient Based Reconstruction Algorithm - GA

Recently proposed gradient algorithm [11] for sparse signal reconstruction is simple algorithm which belongs to the class of gradient CS algorithms [8]. The idea behind this algorithm is to observe missing samples in time domain as the variables, and to adapt their values in a way to produce minimal concentration measure in sparse domain. The main difference of this algorithm compared to others is reconstruction of missing samples in time domain rather than reconstruction of sparse coefficients in transformation (e.g. DFT) domain. The algorithm implementation is given in Algorithm 1 [11].

Algorithm 1 Gradient Algorithm

Require:

- Set of missing/omitted sample positions $\mathbb{N}_A$
- Available samples $y(n)$

1: $x^{(0)}(n) \leftarrow y(n)$ ▷ for $n \in \mathbb{N}_A$
2: $x^{(0)}(n) \leftarrow 0$ ▷ for $n \in \mathbb{N}_Q$
3: $m \leftarrow 0$
4: $\Delta \leftarrow \max|x^{(0)}(n)|$
5: **repeat**
6: **repeat**
7: $x^{(m+1)}(n) \leftarrow x^{(m)}(n)$ ▷ for each n
8: **for** $n_i \in \mathbb{N}_Q$ **do**
9: $X_1(k) \leftarrow \mathrm{DFT}\{x^{(m)}(n) + \Delta\delta(n-n_i)\}$
10: $X_2(k) \leftarrow \mathrm{DFT}\{x^{(m)}(n) - \Delta\delta(n-n_i)\}$
11: $g(n_i) \leftarrow \frac{1}{N}\sum_{k=0}^{N-1} |X_1(k)| - |X_2(k)|$
12: $x^{(m+1)}(n_i) \leftarrow x^{(m)}(n_i) - g(n_i)$
13: **end for**
14: $m \leftarrow m+1$
15: **until** stopping criterion is satisfied
16: $\Delta \leftarrow \Delta/3$
17: **until** required precision is achieved

Output:

- Reconstructed signal $x_R(n) = x^{(m)}(n)$

III. Proposed method

Advantages of both previously mentioned algorithms are combined in order to perform fast and accurate reconstruction, which can be performed with single iteration algorithm, for wide range of applications, which gradient algorithm deals with. The problem of SIRA is detection of signal components in the case when noise caused by missing samples surpass the signal components. Figure Fig.1(a) shows this case. Green circles indicate signal components, while the red cross is used to mark S highest components generally. As we can see signal components and largest components do not match. Therefore, we cannot perform reconstruction with SIRA, since the positions of signal components are not known. This problem can be overcome if we perform few iterations of gradient algorithm. The sparse domain after first, second, and third iteration is shown in Fig.1(b),(c), and (d), respectively. After only three iterations of GA, all signal components become largest in sparse domain (green circles matches red crosses). Now, instead of performing large number of iterations, for examples, 10 (Fig.1(e)) or 40 (Fig.1(f)), with GA in order to remove all noise, a simple and fast reconstruction with SIRA algorithm can be performed. This reconstruction will remove all noise, i.e. reconstruct original signal in just one iteration (3).

However, there is still important task, and it is to detect situation when SIRA performed accurate reconstruction, since the single iteration is performed after each gradient iteration. This issue has been overcome by combining reconstruction properties of two algorithms. Signal reconstructed with SIRA is sparse domain signal $X(k)$. Applying (2) on it, time domain signal $x(n)$ is obtained. Comparing values of available time domain samples, whose positions are from $\mathbb{N}_A$ with the same ones obtained with SIRA algorithm, we can detect situation when they match, and that is the sign that signal is correctly reconstructed with SIRA algorithm.

The pseudo-code for proposed method is given in Algorithm 2. In contrast to standard GA, the lines 7–11 are added in order to detect signal components and perform reconstruction with SIRA, while lines 12–15 are used to check if reconstructed signal matches original one.

IV. Results

In order to check performances of proposed method, a different reconstruction conditions regarding to signal length,

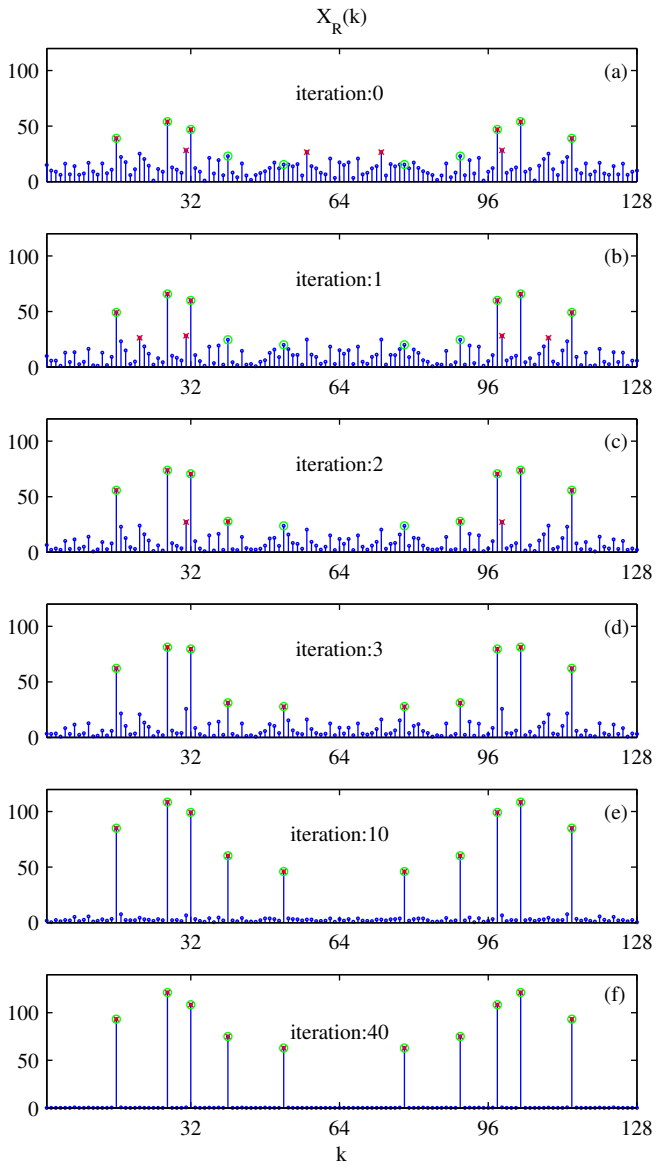

Fig. 1. Single iteration reconstruction performed during gradient alorithm iterations: (a) initial state, (b) after first GA iteration, (c) after second GA iteration, (d) after third GA iteration, (e) after 10th GA iteration, (f) after 40th GA iteration. Green circles indicate positions of nonzero coefficients of original signal. Red marks "x" indicate maximal coeficients of signal to be reconstructed.

sparsity and number of available samples are set in order to compare SIRA, GA and proposed method (GASI). The results are averaged over 50 independent realizations, and are shown in Table I, where: N-signal length, S-signal sparsity, M-number of available samples, SIrec-number of correct reconstructions with SIRA algorithm, GArec-number of correct reconstructions with GA, GA[s]-average time needed for reconstruction with GA, GASIrec-number of correct reconstructions with proposed method,GASI[s]-average time needed for reconstruction with proposed method, Rat.-ratio between averaged

Algorithm 2 Proposed method – GASI

Require:

- Set of missing/omitted sample positions $\mathbb{N}_A$
- Available samples $y(n)$
- Sparsity S

$x^{(0)}(n) \leftarrow y(n)$ ▷ for $n \in \mathbb{N}_A$
$x^{(0)}(n) \leftarrow 0$ ▷ for $n \in \mathbb{N}_Q$
$m \leftarrow 0$
$\Delta \leftarrow \max |x^{(0)}(n)|$
repeat
  repeat
    $X^{(m)}(k) \leftarrow \mathrm{DFT}\{x^{(m)}(n)\}$
    $\mathbf{p} \leftarrow \underset{k_1,k_2,\ldots,k_S}{\operatorname{argmax}} \left|X^{(m)}(k)\right|$
    $\mathbf{A}_{CS} \leftarrow \mathbf{A}(\mathbb{N}_A, \mathbf{p})$
    $X_R(k) = \left(\mathbf{A}_{CS}^T \mathbf{A}_{CS}\right)^{-1} \mathbf{A}_{CS}^T \mathbf{y}$
    $x_R(n) = \mathrm{DFT}^{-1}\{X_R(k)\}$
    if $\sum_{n \in \mathbb{N}_A} \left|x_R(n) - x^{(m)}(n)\right| < 0.01$ **then**
      $x^{(m)}(n) \leftarrow x_R(n)$
      STOP ALGORITHM
    end if
    $x^{(m+1)}(n) \leftarrow x^{(m)}(n)$ ▷ for each n
    for $n_i \in \mathbb{N}_Q$ **do**
      $X_1(k) \leftarrow \mathrm{DFT}\{x^{(m)}(n) + \Delta\delta(n - n_i)\}$
      $X_2(k) \leftarrow \mathrm{DFT}\{x^{(m)}(n) - \Delta\delta(n - n_i)\}$
      $g(n_i) \leftarrow \frac{1}{N} \sum_{k=0}^{N-1} |X_1(k)| - |X_2(k)|$
      $x^{(m+1)}(n_i) \leftarrow x^{(m)}(n_i) - g(n_i)$
    end for
    $m \leftarrow m + 1$
  until stopping criterion is satisfied
  $\Delta \leftarrow \Delta/3$
until required precision is achieved

Output:

- Reconstructed signal $x_R(n) = x^{(m)}(n)$

reconstruction times of GA and proposed method. The average time calculated for GA and GASI is obtained using realizations when observed algorithm performed accurate reconstruction, i.e.

$$\frac{1}{N} \sum_{n=1}^{N} |x(n) - x_r(n)| < 0.01$$

The mean absolute value of 0.01 is considered as good reconstruction. This value can be changed if lower or higher accuracy is needed. Since the SIRA reconstruction could not be performed in the majority of cases, so speed comparison was calculated for GA, and GASI. It is obvious that a significant increase in reconstruction speed is achieved when GASI is used instead of GA.

TABLE I
TABLE SHOWING RECONSTRUCTION PERFORMANCES AVERAGED OVER 50 INDEPENDENT REALIZATIONS, FOR EACH COMBINATION OF N, S, AND M SHOWN IN FIRST THREE COLUMNS.

N	S	M	SI	GA		GASI		Rat.
			rec	rec	[s]	rec	[s]	
128	20	96	42	50	0.06	50	0.053	15
128	16	70	19	50	0.16	50	0.013	29
128	12	50	9	49	0.43	50	0.003	50
128	10	32	4	26	1.22	45	0.004	94
256	40	192	31	50	0.15	50	0.004	17
256	30	140	5	50	0.49	50	0.003	24
256	16	100	16	50	0.74	50	0.003	41
256	8	64	29	50	1.29	50	0.003	76
512	50	384	47	50	0.48	50	0.003	17
512	40	280	16	50	1.31	50	0.003	24
512	30	200	4	50	2.88	50	0.003	27
512	20	128	4	50	6.53	50	0.003	34

TABLE II
AVERAGE NUMBER OF GA ITERATIONS BEFORE THE SINGLE ITERATION RECONSTRUCTION COULD BE PERFORMED.

Number of available samples \ Sparsity S	14	16	18	20	22	24	26	28	30
38	1	1	1	1	2	2	2	2	2
42	1	1	1	2	2	2	2	3	3
46	1	1	2	2	2	2	2	3	3
50	1	2	2	2	3	3	3	4	5
54	1	2	2	2	4	4	4	9	15
58	2	2	3	3	4	6	5	9	15
62	3	3	4	4	7	7	12	25	29
66	3	3	5	8	8	16	27	–	–
70	4	8	10	15	19	41	40	–	–
74	4	9	15	22	40	–	–	–	–
78	6	10	16	29	–	–	–	–	–
82	12	18	–	–	–	–	–	–	–

Example 1: Consider a signal

$$x(n) = \sum_{i=1}^{K} A_i \cos(2\pi k_i n/N + \phi_i)$$

with N = 128. In order to test reconstruction speed of proposed method, sparsity of this signal $S = 2K$ was changed from 2 to N/2, while the randomly chosen amplitudes, frequencies and phases were within $1 \leq A_i \leq 2$, $1 \leq k_i \leq 63$, and $1 \leq \phi_i \leq 2\pi$. Results are averaged over 50 realizations for each combination of sparsity S and number of available samples M. Results shown on Fig.2 presents reconstruction time reduction (reconstruction speed increase) when the proposed method is used instead of standard GA. The reconstructions are up to 80 times faster depending on S and M, when proposed method is used. Table II showing average number of GA iterations which was performed in order to successfully apply single iteration reconstruction. This number is significantly smaller then the number of over 100 iterations which must be performed in order to achieve successful reconstruction when standard GA is used.

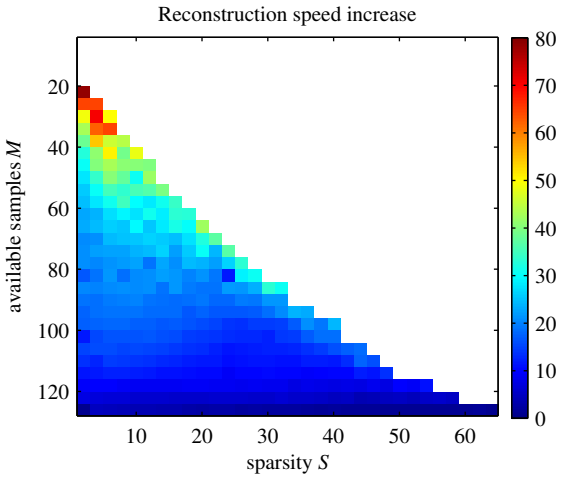

Fig. 2. Reconstruction speed increase as a function of sparsity S and number of available samples M . White color corresponds to the region where reconstruction succes was not above 90% with both of algorithms.

V. CONCLUSION

The algorithm combining gradient based and single iteration reconstruction algorithms is proposed in the paper. Situation when separation of signal components from noise caused by missing samples is not possible is considered. A few iterations of gradient algorithm are used to reduce noise in sparse domain in order to correctly detect signal components, and to perform fast single iteration reconstruction. The proposed method is tested on numerous examples, considering variety of sparsity and number of available samples. It has been shown that proposed method is, depending on condition, 10–100 faster than gradient algorithm, while reconstruction is possible whenever the gradient algorithm can perform reconstruction.

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