Novel Band-Pass and Notch filter with Dynamic Damping of Fractional Order

Marko Bošković, Tomislav B. Šekara, Milan R. Rapaić, Budimir Lutovac, Miloš Daković, Vidan Govedarica

Abstract — **The paper introduces a novel the second order band-pass and notch filter with dynamic damping factor β_d of fractional order. Indeed, the factor β_d has a form of fractional differentiator of order α, i.e. $\beta_d=\beta/s^{\alpha}$, where β and α are adjustable parameters. Shaping of the frequency response enables achieving better phase response compared to the integer-order counterparts which is of great concern in many applications. The aim of the paper is to exploit an extra degree of freedoms of presented filters to achieve the desired filter specifications and obtain desired response in the frequency and the time domain.**

Keywords — **Butterworth filter, Fractional-order filter, Fractional calculus, Frequency response**

I. Introduction

A big number of technical and natural phenomena exhibit a fractional-order (FO) dynamics which by itself lead to a widespread application of fractional calculus (FC) in numerous interdisciplinary fields of science and engineering. FC offers a large exploiting potential since it provides a more accurate models than classical integer-order ones [1-2]. Moreover, the use of fractional differ-integrators (derivatives and integrators) enables characterization of the fractional-order systems with its entire history and modeling non-local and distributed effects. History and fundamental theoretical aspects of FC may be found in [1-5].

The area of application of FC is increasing greatly and rapidly. FC is extensively used in: bioengineering and biomedical applications [6,7], analysis and synthesis of FO electrical elements [8-11], memristive FO systems [12,13], power electronics for FO modeling power converters [14-16], digital image and signal processing [17,18], electromagnetic theory [19,20], time-fractional telegrapher equations for modeling transmission lines [21,22], control systems for designing FO controllers [23-27], mechanics [28,29], diffusion and wave propagation [30-33], nanotechnology, agriculture, economy, etc.

There is a permanent progress in application of fractional calculus in signal (filter) analysis and processing last twenty years. Main application advantage of FO filter is an extra degrees of freedom allowing more precise control of the attenuation slope, which is an efficient feature in biomedical engineering [34,35]. Band and pass-reject (notch) filters have special importance in various engineering applications. Band-pass filters are widely used in wireless transceivers, optical microscopy, seismology etc., while band-stop filters are extensively used in Riemann laser spectroscopy, RF applications etc.

This paper presents a novel the second order band-pass and corresponding notch filter with fractional order dynamic damping factor β_d. It has a form of fractional differentiator of order α, i.e. $\beta_d=\beta/s^{\alpha}$, where fractional order α and adjustable real parameter β are determined to meet specified requirements. For $\alpha=0$ and $\beta=\sqrt{2}$ filter is reduced to a classical the second order filter of Butterworth type. Actually, these parameters are adjusted to obtain desired frequency and time domain response. Shaping the exact frequency response including specified bandwidth is of great concern for many filter applications such as: PLLs (Phase Locked Loops), e.g. in [36] it is of great importance to remove negative phase angle in feedback loop in relay-based critical point estimation, as well as in processing of biomedical signals (ECG, EEG etc.) [37].

This paper is organized as follows. First, a short introduction to the fractional calculus is given in Section 2. Section 3 is briefly on classical filter analysis. In this section novel the second order band-pass and notch filter with fractional damping factor are elaborated. Section 4 gives concluding remarks of the paper.

II. Fundamentals of Fractional Calculus

Fractional order differ-integrator is an operator of fractional calculus which arises from generalization of classical differentiation and integration operators. The transfer function of FO differ-integrator is s^{α} where s is Laplace variable and α is arbitrary real number. For positive α, differ-integrator is a generalization of classical integer order derivative, while for negative α is a generalization of repeated, or n-fold, integral.

Among many others, three most used definitions for the FO derivative and integral operators are Riemann-Liouville, Caputo and Grunwald-Letnikov definitions [3-5]. The left Riemann-Liouville (RL) fractional integral operator of order α is defined as

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$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1}} d\tau, \alpha \in \mathbb{R}, \alpha < 0, \quad (1) $$

where a is a terminal point on interval and $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, z \in \mathbb{C}$ is Euler's Gamma function. RL fractional derivative operator of order α is defined by

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (2) $$

where $n-1 < \alpha < n$. In applications the case $\alpha \in (0,1)$ is of the most importance when equation (2) is reduced to ${}_0D_t^{\alpha} f(t)$ for a=0.

After adopted definition, as an intermediate step for frequency domain signal analysis is calculation of Laplace transform of RL derivative ${}_0D_t^{\alpha} f(t)$. Assuming zero initial conditions Laplace transform of (2) is reduced to

$$ L\left\{ {}_0D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s). \quad (3) $$

III. Analysis of novel band-pass and notch filter with fractional order damping facor

Nowadays, FO filters are growing area of scientific research, so recently different studies of FO filters are conducted. Papers [38,39] introduce FO Butterworth filter dealing with its analysis, active and passive synthesis while design and implementation of FO Butterworth filter for processing biomedical EEG signals is considered in [37]. FO Butterworth low-pass digital filter is designed in [40] for sharpening a digital image which quality is adjusted through changing the FO of the filter. However, faster roll-off may be achieved, e.g. with Chebyshev filter at the expense of ripples in pass and stop bands [41], so in [42] is developed a complex FO low-pass filter.

The most common types of analog filter types are the Butterworth, Chebyshev (I and II), Bessel and Elliptic. Let set aside a Butterworth filter which characterizes with a maximally flat response with no ripple compared to the others. Magnitude frequency response rolls-off smoothly and monotonic, with a low-pass or highpass roll off 20dB/dec for every pole. Thus, a third order Butterworth band-pass filter would have an attenuation rate of -60dB/dec and 60 dB/dec. Classical integer-order analog Butterworth filter of order n has frequency magnitude response [43]

$$ |F(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \quad (4) $$

where ω_c is the 3 dB cut-off frequency. For example, if we select n=2 and n=3, corresponding transfer function of such filter with response (4) are, respectively

$$ F(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}, \quad (5) $$

$$ F(s) = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}. \quad (6) $$

The most common filter structures are those based on the analog second-order filter. Hence, in this paper the second order band-pass filter is defined with normalized transfer function

$$ F_{bp}(s) = \frac{\beta_d s}{s^2 + \beta_d s + 1} \quad (7) $$

where $\beta_d=\beta/s^{\alpha}$ is dynamic damping parameter of fractional order, α is a fractional-order parameter, and β is real adjustable parameter, in general case independent of α, used to meet specified requirements and ω_c=1 s^{-1} is a normalized cut-off frequency. By substituting s with s/ω_c in (8), filter can be designed for any other cut-off frequency ω_c. On the basis of (8), the corresponding notch filter transfer function is defined as $F_n(s)=1-F_{bp}(s)$, i.e.

$$ F_n(s) = \frac{s^2 + 1}{s^2 + \beta_d s + 1} \quad (8) $$

It is obvious that for $\alpha = 0$ and $\beta = \sqrt{2}$ Eq. (7) and (8) are reduced to classical integer-order band-pass and notch filter of Butterworth type, respectively, with the same characteristic equation of low-pass Butterworth filter in Eq. (5).

In order to improve phase response and not deteriorate a magnitude response, parameter β is determined following the idea in [44] to keep the same dominant dynamics which is determined with characteristic equation in Eq. (7) and (8). First, the overshoot A_p=4.32% in the unit-step response of classical the second-order low-pass Butterworth filter $F_{lp}(s) = 1/(s^2 + \sqrt{2}s + 1)$ is calculated. Then, the unit-step response of FO low-pass counterpart of filters (7) and (8)

$$ F_{lp}(s) = \frac{1}{s^2 + \beta_d s + 1} \quad (9) $$

is determined via numerical inversion of Laplace transform which enables to calculate β to keep the same A_p for different values of fractional order α.

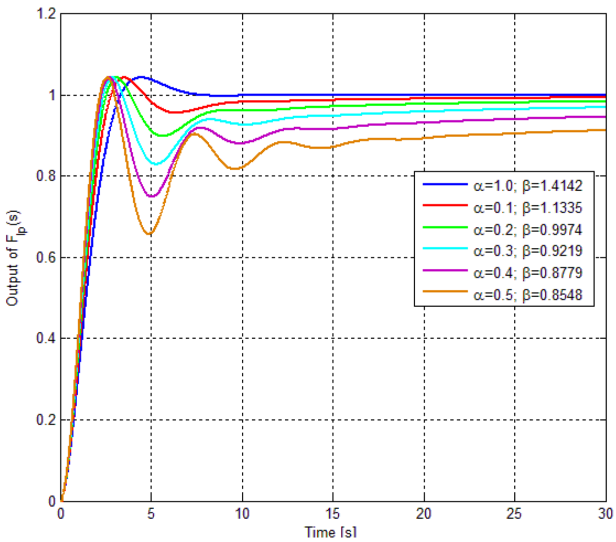

Fig. 1. Unit-step response of low-pass filter defined in Eq. (10) for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. and specific β to obtain the same overshoot A_p=4.32%

Obtained unit-step response of filter (9) for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. are shown in Fig. 1, and calculated values of β are given in Table I.

TABLE 1: VALUES OF ADJUSTABLE PARAMETER β TO MEET SPECIFIED REQUIREMENTS IN OVESRHOOT

α	β
0	$\sqrt{2} \approx 1.4142$
0.1	$0.8015\sqrt{2} \approx 1.1335$
0.2	$0.7058\sqrt{2} \approx 0.9982$
0.3	$0.6519\sqrt{2} \approx 0.9219$
0.4	$0.6208\sqrt{2} \approx 0.8779$
0.5	$0.6044\sqrt{2} \approx 0.8547$

The applied idea actually leads to saving the same bandwidth of the systems since dynamics of low-pass, band-pass and notch filters in Eq. (7)-(9) is determined by roots of the same characteristic equation $s^2+\beta s^\alpha+1$, which are shown in complex s-plane in Fig. 2 for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$.

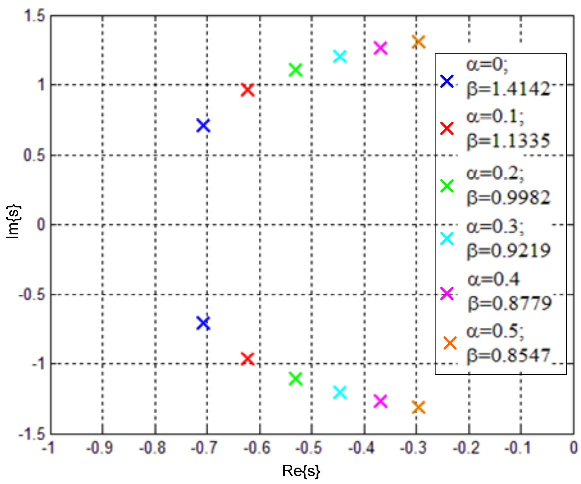

Fig. 2. Roots in s-plane of characteristic equation of $F_{bp}(s)$ and $F_n(s)$for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4; 0.5\}$. to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1

Values of magnitude and phase for both introduced filters, band-pass and notch, at important frequencies are given in Table II.

TABLE II: MAGNITUDE AND PHASE VALUES AT IMPORTANT FREQUENCIES OF THE PROPOSED FO BAND-PASS AND NOTCH FILTER

ω	$\lvert F_{bp}(j\omega)\rvert$	$\angle F_{bp}(j\omega)$	$\lvert F_n(j\omega)\rvert$	$\angle F_n(j\omega)$
$\omega \to 0$	0	$\frac{(1-\alpha)\pi}{2}$	1	0
$\omega = \omega_c$	1	0	0	$(\alpha-1)\frac{\pi}{2}$ for $\omega \to \omega_{c-}$ $(\alpha+1)\frac{\pi}{2}$ for $\omega \to \omega_{c+}$
$\omega \to \infty$	0	$-\frac{(\alpha+1)\pi}{2}$	1	0

Magnitude and phase frequency responses of FO band-pass filter in Eq. (7) with damping factor $\beta_d = \beta / s^\alpha$ in Table I are shown in Fig. 3.

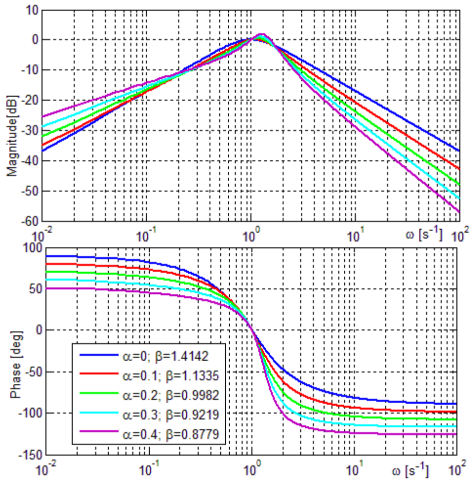

Fig. 3. Magnitude and phase frequency characteristics of fractional-band pass filter $F_{bp}(s)$ for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4\}$. and specific β to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1.

Bandwidth of band-pass filter in Eq. (9) shown in Fig. 3 for $\alpha = 0$ is defined with lower cut-off frequency $\omega_{cl} \approx 0.52\ s^{-1}$ and upper frequency $\omega_{c2} \approx 1.93\ s^{-1}$. Magnitude and phase frequency responses of FO notch filter in Eq. 8) with damping factor $\beta_d = \beta / s^\alpha$ in Table I are shown in Fig. 4.

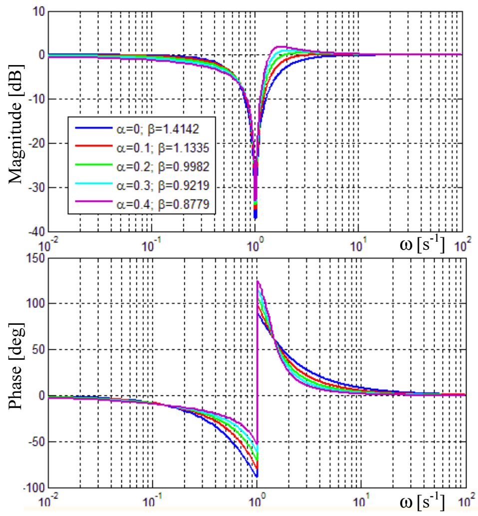

Fig. 4. Magnitude and phase frequency characteristics of fractional-band notch filter $F_n(s)$ for $\alpha \in \{0; 0.1; 0.2; 0.3; 0.4\}$. and specific β to obtain the same overshoot A_p=4.32% of $F_{lp}(s)$ in Fig. 1.

As it can be seen from Figs. 3 and 4, additional flexibility is supported with use of presented band-pass and notch filter with FO damping factor. By choosing an FO parameter it is enabled to adjust band-pass/band-reject and to decrease a large negative phase for notch filter which is important in some applications as in system identification [36]. Indeed, there are increasing number of designs of FO filters with possibility to adjust and shape desired frequency response, e.g. in [45] is reported an electronic way of control of FO order and pole frequency of low-pass filter through adjustment of current gain of current amplifiers.

IV. Conclusion

The second order band-pass and notch filters with dynamic damping factor of fractional-order are introduced and analyzed in this paper. The existence of fractional-order parameter enables more precise and flexible shaping of the frequency responses of both filters which is of great importance in big number of applications.

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