

Additive Noise Influence on the Bivariate Two-Component Signal Decomposition

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Abstract—Decomposition of multicomponent signals overlapping in the time-frequency domain is a challenging research topic. To solve this problem, many approaches have been proposed so far, but only to be efficient for some particular signal classes. Recently, we have proposed a decomposition approach for multivariate multicomponent signals, based on the time-frequency signal analysis and concentration measures. The proposed solution is efficient for multivariate signals partially overlapped in the time-frequency plane regardless of the non-stationarity type of particular signal components. This decomposition approach is shown to be also efficient in noisy environments. In this paper, we investigate the limits of the decomposition efficiency subject to the signal-to-noise ratio and initial phase differences between the signals from different channels. The paper is focused on the decomposition of bivariate two-component signals.

Keywords— Concentration measures; multivariate signals; signal decomposition; time-frequency signal analysis

I. INTRODUCTION

Time-frequency signal analysis deals with signals having a time-varying spectral content [1]– [8]. The analysis and processing of such signals, widely known as non-stationary, is typically difficult using the classical Fourier analysis [7]. Multichannel signals, a form of multivariate data, arise through recent sensor technology developments [2]. The processing of these signals is an ongoing research topic [1], [2], [9]–[12].

In the classical time-frequency analysis, several approaches for the decomposition of multicomponent signals have been proposed [3]–[6]. Except for some special signal forms, extraction of signal components overlapped in the time-frequency plane is not possible [1], [2]. However, using the potential of the multivariate signal form, we have recently developed an algorithm for the decomposition of multicomponent signals, leading to a complete extraction of components partially overlapped in the time-frequency plane [1], [2]. The basic idea is drawn from an S-method based decomposition approach, originally proposed in [3]. The robustness of the decomposition method [1], [2] on the influence of the noise is highly related with the initial phase difference between variates, that is, signals from each channel. In this paper, we investigate

this dependence, by checking numerically which range of initial phase difference between variate produces a successful decomposition, for given signal-to-noise ratio (SNR).

The paper is organized as follows. After Introduction, the basic theory is introduced in Section II. The decomposition of bivariate two-component signals is presented in Section III. The influence of additive Gaussian noise on the decomposition is numerically analyzed in the same section. The paper ends with concluding remarks.

II. BASIC THEORY

A bivariate signal can be defined as

$$\mathbf{x}(t) = \begin{bmatrix} a_1(t)e^{j\phi_1(t)} \\ a_2(t)e^{j\phi_2(t)} \end{bmatrix}, \quad (1)$$

and it can be interpreted as a signal obtained measuring a physical process $x(t)$ by two sensors, with $a_1(t)\exp(j\phi_1(t)) = \chi_1 x(t)\exp(j\varphi_1)$ for the first sensor channel and $a_2(t)\exp(j\phi_2(t)) = \chi_2 x(t)\exp(j\varphi_2)$ for the second one. Without loss of generality, it is assumed that each sensor modifies amplitude and phase of the measured signal.

The Wigner distribution (WD) of bivariate signal $\mathbf{x}(t)$ is a time-frequency representation with the following form

$$\begin{aligned} WD(\omega, t) &= \int_{-\infty}^{\infty} \mathbf{x}^H(t - \frac{\tau}{2})\mathbf{x}(t + \frac{\tau}{2})e^{-j\omega\tau} d\tau \\ &= \sum_i \int_{-\infty}^{\infty} a_i^*(t - \frac{\tau}{2})a_i(t + \frac{\tau}{2})e^{j(\phi_i(t+\frac{\tau}{2})-\phi_i(t-\frac{\tau}{2}))}e^{-j\omega\tau} d\tau \end{aligned}$$

where $\mathbf{x}^H(t)$ is the Hermitian transpose of vector $\mathbf{x}(t)$ and $i = 1, 2$. In the case of a two-component bivariate signal

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_1(t) + \mathbf{x}_2(t) \\ &= \begin{bmatrix} \chi_{11}x_1(t)e^{j\varphi_{11}} \\ \chi_{12}x_1(t)e^{j\varphi_{12}} \end{bmatrix} + \begin{bmatrix} \chi_{21}x_2(t)e^{j\varphi_{21}} \\ \chi_{22}x_2(t)e^{j\varphi_{22}} \end{bmatrix}, \quad (2) \end{aligned}$$

where $x_i(t) = A_i(t)e^{j\psi_i(t)}$, $i = 1, 2$, the Wigner distribution takes the form

$$WD(\omega, t) = WD_a(t, \omega) + WD_c(t, \omega). \quad (3)$$

Here, $WD_a(t, \omega)$ is used to denote the auto-terms

$$WD_a(\omega, t) = \int_{-\infty}^{\infty} [\chi_{11}^2 + \chi_{12}^2] x_1^*(t - \frac{\tau}{2}) x_1(t + \frac{\tau}{2}) e^{-j\omega\tau} d\tau \\ + \int_{-\infty}^{\infty} [\chi_{21}^2 + \chi_{22}^2] x_2^*(t - \frac{\tau}{2}) x_2(t + \frac{\tau}{2}) e^{-j\omega\tau} d\tau$$

whereas $WD_c(t, \omega)$ is used to denote the undesirable cross-terms. Using the definition (2), this term is further expanded

$$WD_c(\omega, t) = \int_{-\infty}^{\infty} [\chi_{11}\chi_{21}x_1^*(t - \frac{\tau}{2})x_2(t + \frac{\tau}{2})e^{j(\varphi_{21}-\varphi_{11})} \\ + \chi_{11}\chi_{21}x_2^*(t - \frac{\tau}{2})x_1(t + \frac{\tau}{2})e^{j(\varphi_{11}-\varphi_{21})} \\ + \chi_{12}\chi_{22}x_1^*(t - \frac{\tau}{2})x_2(t + \frac{\tau}{2})e^{j(\varphi_{22}-\varphi_{21})} \\ + \chi_{12}\chi_{22}x_2^*(t - \frac{\tau}{2})x_1(t + \frac{\tau}{2})e^{j(\varphi_{12}-\varphi_{22})}] e^{-j\omega\tau} d\tau.$$

The auto-terms $WD_a(t, \omega)$ are summed up in phase, as phase shifts cancel out. On contrary, in the case of cross-terms $WD_c(t, \omega)$, phase shifts do not cancel out and it is expected that the bivariate signal form leads to the cross-terms cancellation as these terms average out, when compared with the univariate signal WD.

It is important to note that in (2) it is assumed that components $x_1(t) = A_1(t)e^{j\psi_1(t)}$ and $x_2(t) = A_2(t)e^{j\psi_2(t)}$ of the measured signal exhibit slow-varying amplitude changes compared to phase-changes, i.e. $|d\chi_{ij}(t)/dt| \ll |d\psi_i(t)/dt|$. Therefore, the amplitudes may be considered as constant within the observed time interval, $\chi_{ij}(t) \sim \chi_{ij}$. The successful cancellation of the cross-terms is dependent on the phase differences $\varphi_{ij} - \varphi_{mn}$, where $i, j, m, n = 1, 2$. In the noiseless signals case, even very small phase differences $\varphi_{i,j} - \varphi_{m,n} \ll 1$ lead to a successful cancellation of undesirable cross-terms.

III. SIGNAL DECOMPOSITION

After the proper discretization, the inverse discrete Wigner distribution can be written in the following form

$$\mathbf{x}^H(n_2)\mathbf{x}(n_1) = \\ = \frac{1}{K+1} \sum_{k=-K/2}^{K/2} WD\left(\frac{n_1+n_2}{2}, k\right) e^{j\frac{\pi}{K+1}k(n_1-n_2)}. \quad (4)$$

where n_1 and n_2 are proper discrete-time indexes obtained after discretization of t and $\tau/2$ with a sampling period Δt and applying proper substitutions [1], [2]. Assuming that the cross-terms can be neglected, based on the WD autocorrelation function, we form matrix \mathbf{R} with elements

$$R(n_1, n_2) = \mathbf{x}^H(n_2)\mathbf{x}(n_1) \\ = [\chi_{11}^2 + \chi_{12}^2]x_1(n_1)x_1^*(n_2) + [\chi_{12}^2 + \chi_{22}^2]x_2(n_1)x_2^*(n_2).$$

A. Decomposition of a two-component bivariate signal

Standard eigenvalue decomposition of the square matrix \mathbf{R} of dimensions $K \times K$ leads to

$$\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T = \sum_{p=1}^K \lambda_p \mathbf{q}_p(n) \mathbf{q}_p^H(n), \quad (5)$$

where λ_p are eigenvalues and $\mathbf{q}_p(n)$ are eigenvectors of \mathbf{R} . Note that the eigenvectors $\mathbf{q}_p(n)$ are orthonormal. For a two-component signal, in a noiseless case, the elements of this matrix are

$$R(n_1, n_2) = \lambda_1 q_1(n_1) q_1^*(n_2) + \lambda_2 q_2(n_1) q_2^*(n_2). \quad (6)$$

Although the eigenvectors are mutually orthogonal, the overlapped signal components are not orthogonal. Therefore, both eigenvectors \mathbf{q}_1 and \mathbf{q}_2 contain a linear combination of signal components. This means that each component can be expressed as a linear combination of eigenvectors \mathbf{q}_1 and \mathbf{q}_2 :

$$\mathbf{x}_1 = k_{11}\mathbf{q}_1 + k_{12}\mathbf{q}_2, \quad (7)$$

$$\mathbf{x}_2 = k_{21}\mathbf{q}_1 + k_{22}\mathbf{q}_2. \quad (8)$$

Notice that individual signal components are better concentrated in the time-frequency plane than their linear combinations contained within eigenvectors \mathbf{q}_1 and \mathbf{q}_2 . Therefore, the basic idea for the decomposition is to search for the unknown coefficients k_{ij} , $i, j = 1, 2$ which produce the best possible individual component concentrations. The procedure for finding unknown coefficients k_{ij} is presented in Algorithm 1. A detailed description of the procedure can be found in [1]. The concentration measure used in step 7 is defined by

$$\mathcal{M}[TFR_{\mathbf{y}}(n, k)] = \sum_n \sum_k |TFR_{\mathbf{y}}(n, k)|, \quad (9)$$

for a vector \mathbf{y} , and it is the concept borrowed from both sparse signal processing and compressed sensing frameworks, as well as from the classical time-frequency analysis [8]. To solve the optimization problem in step 7 we use the minimization procedure based on the direct search. For bivariate signal, $P = 2$ is used.

B. Noisy signal decomposition

The efficiency of the presented procedure in noise environment and for different channel phases will be tested next. Consider real noisy bivariate signal $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ of the form

$$x_i(t) = e^{-(t/128)^2} \cos\left(\frac{300\pi t}{64 \cdot 128} + \varphi_i\right) + \epsilon_i(t) \\ = 0.5e^{-(t/128)^2} \left[e^{j\left(\frac{300\pi t}{64 \cdot 128} + \varphi_i\right)} + e^{-j\left(\frac{300\pi t}{64 \cdot 128} + \varphi_i\right)} \right] + \epsilon_i(t) \\ = x_{1i}(t) + x_{2i}(t) + \epsilon_i(t), \quad i = 1, 2, \quad (10)$$

Algorithm 1 Multivariate signal decomposition**Input:**

- Multivariate signal $\mathbf{x}(n)$

1: Calculate elements of matrix \mathbf{R} as

$$R(n_1, n_2) = \mathbf{x}^H(n_2)\mathbf{x}(n_1).$$

2: Find eigenvectors \mathbf{q}_i and eigenvalues λ_i of matrix \mathbf{R} .

3: $P \leftarrow$ number of non-zero eigenvalues

4: **repeat**

5: $N_{up} \leftarrow 0$

6: **for** $i = 1, \dots, P$ **do**

7: Solve minimization problem

$$\min_{k_1, \dots, k_P} \mathcal{M} \left\{ \text{TFR} \left\{ \frac{1}{C} \sum_{p=1}^P k_p \mathbf{q}_p \right\} \right\} \text{ s. t. } k_i = 1$$

where the normalization of the combined signal to

1 is done using $C = \sqrt{\left\| \sum_{p=1}^P k_p \mathbf{q}_p \right\|_2}$

8: **if** any $k_p \neq 0, p \neq i$ **then**

9: $\mathbf{q}_i \leftarrow \frac{1}{C} \sum_{p=1}^P k_p \mathbf{q}_p$

10: **for** $k = i + 1, i + 2, \dots, P$ **do**

11: $\mathbf{q}_k \leftarrow \frac{1}{\sqrt{1 - |\mathbf{q}_i^H \mathbf{q}_k|^2}} (\mathbf{q}_k - \mathbf{q}_i^H \mathbf{q}_k \mathbf{q}_i)$

12: **end for**

13: $N_{up} \leftarrow N_{up} + 1$

14: **end if**

15: **end for**

16: **until** $N_{up} = 0$

Output:

- Reconstructed signal components $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_P$

defined for $-128 \leq t \leq 128$. One realization of the signal is shown in Fig. 1 (a), for the first channel and SNR=10dB. In the considered experiment, the phases $\varphi_1 \neq \varphi_2$ are random numbers with a uniform distribution drawn from the interval $[0, 2\pi]$. In the experiment, we test the influence of the phase difference $\nabla = \varphi_1 - \varphi_2$ on the successful decomposition. For signal in Fig. 1 this difference is 80 degrees. The considered signal is real-valued, with two symmetric components $x_{1i}(t)$ and $x_{2i}(t)$ existing in the Fourier transform and the time-frequency domains. These components partially overlap, and thus they are inseparable using these representations. This is illustrated in Fig. 2, where neither the PWD of the analytic signal (a), nor the PWD of the given signal (b) produce desirable results. Bedrosian's product theorem condition for amplitude and phase is not satisfied in this case. The channel noises $\epsilon_i(t)$ are independent, real, Gaussian, zero-mean white noises, with equal variances σ^2 . The decomposition of the

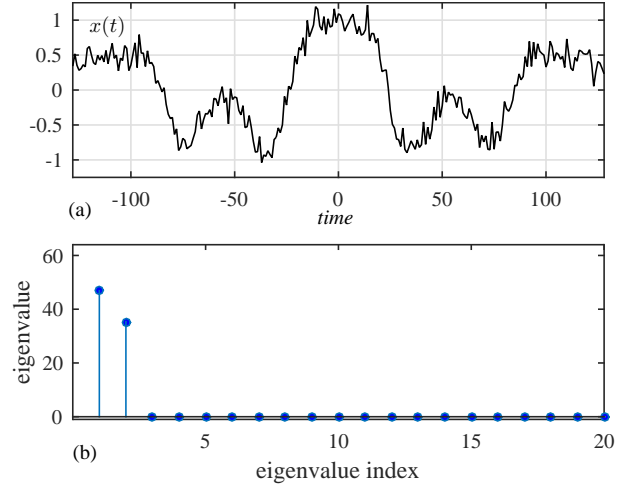


Fig. 1. The analyzed bivariate real signal. (a) time domain waveform; (b) Eigenvalues of the autocorrelation matrix \mathbf{R} ; (c) Instantaneous frequency estimation of the signal.

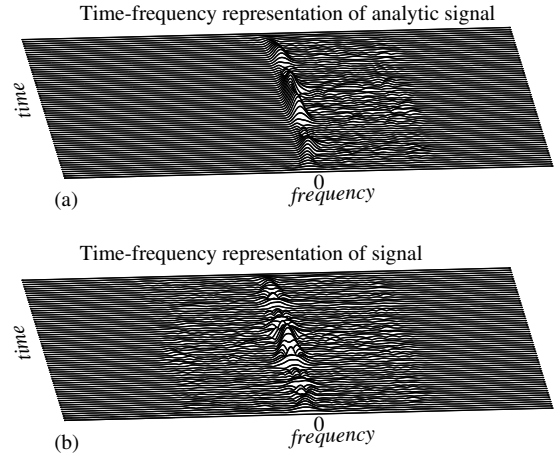


Fig. 2. Time-frequency representation of the considered bivariate signal. (a) Pseudo-WD of the analytic signal; (b) Pseudo-WD of the original signal

signal is done using the procedure presented in Algorithm 1. In this experiment we vary the phase difference $\nabla = \varphi_1 - \varphi_2$ between bivariate components, from 0 to 90 degrees. The bivariate signal was corrupted with additive white Gaussian noises with zero mean value and variances such that five SNR levels are obtained: 10dB, 20dB, 30dB, 40dB and 50dB. For each considered ∇ and SNR level, the probability of successful decomposition is calculated averaging results based on 30 independent realizations of noisy signal. Results are presented in Fig. 5, clearly indicating that the presented decomposition method is quite robust on the noise influence. Larger differences ∇ lead to better extraction results. In all cases, the best results are obtained for $\nabla = \pi/2$ (90 degrees). The time-frequency representations (Pseudo-WD) of the eigenvectors corresponding to the most significant eigenvalues (these eigen-

vectors are linear combinations of signal components) of the autocorrelation matrix \mathbf{R} are shown in Fig. 3. The Pseudo-WD of successfully extracted signal components is shown in Fig. 4.

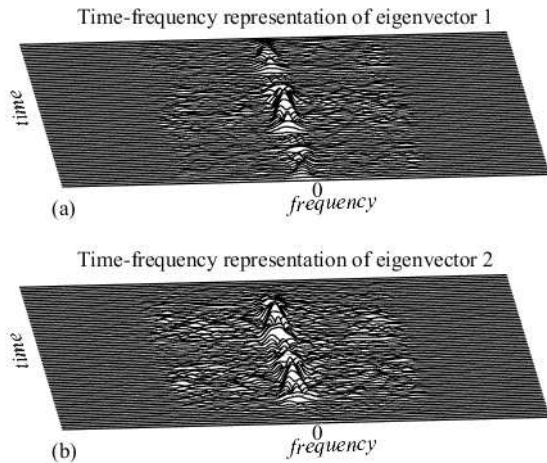


Fig. 3. Pseudo-WD of first two eigenvectors of matrix \mathbf{R} .

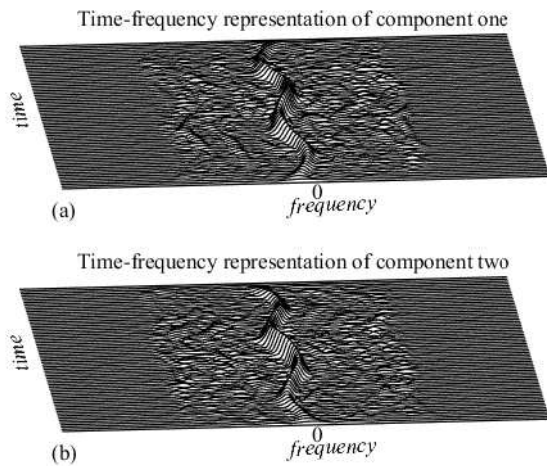


Fig. 4. Pseudo-WD of successfully extracted signal components.

IV. CONCLUSION

The additive noise influence on the decomposition of bivariate two-component signals is analyzed. It is numerically shown that the decomposition success is dependent on the phase difference between components from each channel. The results indicate the robustness of the decomposition algorithm

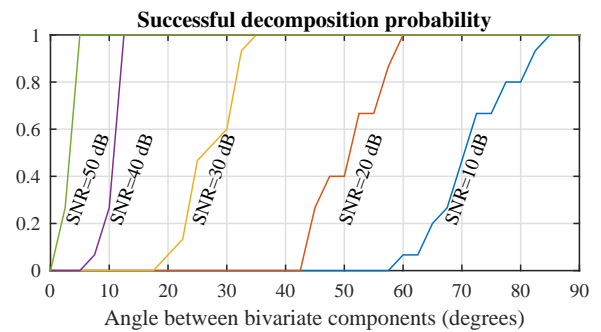


Fig. 5. The successful decomposition probability shown for different angles between bivariate components and for different SNR levels.

on the noise influence. The results are dependent on angles between signals in different channels. Our further research will be oriented towards more detailed statistical analysis of the method in noisy environments.

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