# Sparse Signal Reconstruction in Dual Polynomial Fourier Transform

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***Abstract*—The reconstruction of sparse acoustic waves transmitted through a dispersive channel is examined. When a signal propagates through such a channel, it changes its characteristics and produces new components in the received part. In real-world signals, some disturbances are introduced during the transmission. Common forms of disturbances are strong sinusoidal signals from various sources. Frequency components with such disturbances can be filtered out. Since the signal components can be considered as sparse in the the dual polynomial Fourier transform (DPFT) domain, the removed spectral component can be reconstructed using compressive sensing methods. To this aim, a method for decomposition and reconstruction of multicomponent signals in dispersive environment is introduced. The method is based on decomposition of parameters using DPFT and their reconstruction from a reduced set of signal spectral components using compressive sensing framework.**

***Keywords*—compressive sensing; dispersive channels; polynomial Fourier transform; sparsity; time-frequency analysis***

## I. Introduction

Underwater acoustic channels are known for their dispersive characteristic. The problem of dispersivity has been a challenge in their analysis. A dispersive channel changes the characteristics of signals during their transmission. The main problems of dispersive channels is producing nonlinear transformations of the signal, shifts in phase of the signal and adding new components due to multipath propagation [1]–[4].

Other problem of the dispersive channels is that the signals become complex and non-stationary. This is the reason why non-stationary signal processing theory is suitable in the analysis of those signals. The most common tool for the analysis of non-stationary signals is the time-frequency signal analysis [5]–[7]. Common problem in practice are strong harmonic disturbances. After these disturbances are removed, the signal components should be reconstructed. Various localization techniques for dispersive channels were developed, such as using the phase continuity in [2] or using narrowband systems with unitary warping relations in [4].

In the theory of sparse signal reconstruction, a signal is sparse if it has only few non-zero components in comparison to the total length of the signal. If the signal is sparse, it can be reconstructed with less measurements [8]–[12]. The considered acoustic signal is sparse in the dual polynomial Fourier transform (DPFT) domain, and the noisy measurements (impulses) occur in frequency domain. The impulses in frequency domain will introduce new sinusoids in time domain. These disturbances are removed, and the signal components can be reconstructed by compressive sensing methods, such as the matching pursuit algorithm.

The paper is organized as follows. In Section II the noisy received signal is modelled, and in Section III basic theory of compressive sensing is introduced. In Section IV we define the DPFT domain and its sparsity conditions as in sparse signal reconstruction framework. An example of the decomposition and reconstruction is be shown in Section V, and the conclusions are presented in VI.

## II. Received Signal Modelling

Assume a linearly frequency modulated (LFM) signal of the form

$$u(n) = e^{j\pi\alpha n^2} \tag{1}$$

is transmitted. The signal propagates through an underwater dispersive channel. The channel considered is an iso-velocity channel [2], having the same velocity of sound over all volume [1]–[4]. The transfer function of the channel is

$$H(f) = \sum_{m=1}^{+\infty} A_t(m,f,r)\exp\big(jk_r(m,f)r\big) \tag{2}$$

where $m$ is the mode index, $r$ is the distance between the transmitter and received, $A_t(m,f,r)$ is the attenuation rate and $k_r(m,f)$ are the horizontal wavenumbers. The attenuation rate dependence on the distance $r$ is $A_t(m,f,r) = A(m,f)/\sqrt{r}$.

The transfer function depends on the number of the modes, and the modes are dependent on wavenumbers [2]

$$k_r(m,f) = \Big(\frac{2\pi f}{c}\Big)^2 - \Big((m-0.5)\frac{\pi}{D}\Big)^2 \tag{3}$$

where $D$ is the channel depth, and $c = 1500$m/s is sound speed. The received signal is then

$$x(n) = u(n) * h(n), \tag{4}$$

where $h(n)$ is the impulse response of (2). Our goal is to decompose the mode functions, which will make the problem of detecting the transmitted signal straightforward. This decomposition makes compressive sensing methods application possible to use as well. The decomposition method will be formulated within the compressive sensing approach.

## III. Signal Decomposition and Reconstruction

Assume a signal $x(n)$, $0 \leq n < N$ and its linear transform $X(k)$. In the vector form they are written as

$$\mathbf{x} = [x(0), x(1), \ldots, x(N-1)]^T \tag{5}$$

$$\mathbf{X} = [X(0), X(1), ..., X(N-1)]^T. \tag{6}$$

They are related via $N \times N$ transformation matrix $\mathbf{A}_N$ as

$$\mathbf{X} = \mathbf{A}_N \mathbf{x}. \tag{7}$$

We will assume that signal $x(n)$ is sparse. It means that the signal $\mathbf{x}$ has only $K \ll N$ samples $x(n_1), x(n_2), \ldots, x(n_K)$ that are non-zero. When the signal is sparse in one of its domains, it can be reconstructed with less measurements in one of its transformation domains, i.e. with $N_A < N$. The signal measurements in this case are coefficients of its transform at positions $\mathbb{N}_{\mathbb{A}} = \{k_1, k_2, \ldots, k_{N_A}\}$. The measurement vector is defined by

$$\mathbf{y} = [X(k_1), X(k_2), \ldots, X(k_{N_A})]^T. \tag{8}$$

Vector form of the measurements equation is

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{9}$$

where $\mathbf{A}$ is a $N_A \times N$ partial transform matrix obtained by keeping only the rows of $\mathbf{A}_N$ corresponding to the available measurements. The compressive sensing states that the signal can be reconstructed, if the reconstruction conditions are met, by minimizing $\mathbf{x}$ using the available measurements $\mathbf{y}$, i.e.

$$\min \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{y} = \mathbf{A}\mathbf{x}. \tag{10}$$

The solution of problem (10) can be found in various ways. One of the common algorithms to solve the problem is the orthogonal matching pursuit (OMP) [12]. In the first step of the OMP, the position of the largest component is found

$$n_1 = \arg\max\{\mathbf{x}_0\} \tag{11}$$

using the inital estimate $\mathbf{x}_0 = \mathbf{A}^H \mathbf{y}$, calculated using only the available measurements. A new partial matrix of the matrix $\mathbf{A}$ is formed, omitting all rows except the row which corresponds to the estimated position $n_1$. New matrix is then $\mathbf{A}_1$. The estimate of the first component in the time domain is

$$\mathbf{x}_1 = (\mathbf{A}_1^H \mathbf{A}_1)^{-1} \mathbf{A}_1^H \mathbf{y}. \tag{12}$$

The signal is reconstructed at the position $n_1$ and subtracted from the original signal measurements. The estimate of the non-zero position is calculated again with this signal and its maximum position is found at $n_2$. A new set $\mathbb{K} = \{n_1, n_2\}$ is formed with the corresponding matrix $\mathbf{A}_2$. The new estimate $\mathbf{x}_2$ is calculated and the signal is reconstructed. The procedure is repeated until all $K$ components are reconstructed. For the case when the signal samples are spread, we may use few instances around $n_i$ in each reconstruction step.

We will assume that the signal is $K$-sparse in the time domain, and we will consider dual polynomial transform coefficients as measurements.

## IV. Dual Polynomial Fourier Transform Sparsity

Localization using dual polynomial Fourier transform (DPFT) was studied in [13]. It is of crucial importance to obtain a sparse domain representation of the signal. In this paper, we will consider that there are some unavailable coefficients in the frequency domain (due to denoising procedure on harmonic disturbances). The combination of DPFT with the OMP algorithm for a successful reconstruction and decomposition of the signal samples will be presented.

### A. Definition of DPFT

The main goal is to find the parameters where the DPFT of a signal produces the highest concentration, meaning maximal sparsity. Then we can extract and localize the positions of the components [7], [13], [14].

The considered signal is a polynomial phase signal

$$X(k) = Ae^{-j\sum_{l=1}^{N} b_l k^l}. \tag{13}$$

in the frequency domain. This is why the most suitable tool for analysis is the DPFT, instead of the classical PFT. The discrete DPFT is defined as

$$x_{\beta_2,\beta_3\ldots,\beta_N}(n) = \sum_k X(k) e^{j\frac{2\pi}{N}(nk+\beta_2 k^2+\ldots+\beta_N k^N)}. \tag{14}$$

The signal will be highly concentrated when the maximum of the transform is matched. It is achieved when

$$(\hat{b}_1, \hat{b}_2, ..., \hat{b}_N) = \arg \max_{(n,\beta_2,\ldots,\beta_N)} |x_{\beta_2,\ldots,\beta_N}(n)|. \tag{15}$$

Ideally, the best DPFT concentration is when $(\beta_2, ..., \beta_N) = (b_2, ..., b_N)$. Our goal is to estimate the parameters such that $\hat{b}_2 \approx b_2, \ldots, \hat{b}_N \approx b_N$.

In some real-world scenarios, the signal will be received with a kind of disturbance. Here, we will assume that the signal is corrupted with strong sinusoidal disturbances

$$x_d(n) = x(n) + \sum_{l=1}^{N_M} B_i e^{j(\omega_l n + \psi_l)}. \tag{16}$$

The strong periodic disturbances can be detected and removed by a notch filter. Then a reduced set of frequencies $k \in \mathbb{N}_{\mathbb{A}}$ is available in $X(k)$. The reconstruction is presented next.

### B. Sparsity in DPFT

We will restrict the analysis to the third polynomial order. Consider that $\mathbf{X}$ has disturbed samples which are found and set as unavailable. The third order DPFT estimated using only the available samples of $\mathbf{X}$ is

$$x_{\beta_2,\beta_3}(n) = \sum_{k\in\mathbb{N}_{\mathbb{A}}} X(k)e^{j\frac{2\pi}{N}(nk+\beta_2k^2+\beta_3k^3)} \tag{17}$$

for

$$X(k) = Ae^{-j\frac{2\pi}{N}(b_1k+b_2k^2+b_3k^3)}. \tag{18}$$

Assume that parameters $\beta_2, \beta_3$ are found by a direct search over the interval of their possible values. When the parameters are correctly estimated $(\beta_2, \beta_3) = (b_2, b_3)$, the DPFT is

$$x_{b_2,b_3}(n) = \sum_k Ae^{j\frac{2\pi}{N}n(k-b_1)} = A\delta(k-b_1). \tag{19}$$

It is sparse. In the case of multicomponent signals, i.e.

$$X(k) = \sum_{m=1}^{M} A_m e^{-j(b_{1m}k+b_{2m}k^2+b_{3m}k^3)}, \tag{20}$$

the parameters of components are estimated in iterative way. Without loss of generality, assume that $A_1 > A_2 > \cdots > A_M$. When the first component is matched with

$$(\beta_{21}, \beta_{31}) = (b_{21}, b_{31})$$

we may consider that all other components are spread and negligible. The measurements matrix is obtained from this signal definition assuming that only the values $k \in \mathbb{N}_{\mathbb{A}}$ are available

$$x_{b_{21},b_{31}}(n) = \sum_{k\in\mathbb{N}_{\mathbb{A}}} X(k)e^{j(b_{21}k^2+b_{31}k^3)}e^{j\frac{2\pi}{N}n(k-b_{11})}. \tag{21}$$

This relation for various $n$ can be written as

$$\begin{bmatrix} x_{b_{21},b_{31}}(n_1) \\ x_{b_{22},b_{32}}(n_2) \\ \vdots \\ x_{b_{2K},b_{3K}}(n_K) \end{bmatrix} = \mathbf{A}_K^H \begin{bmatrix} X(k_1) \\ X(k_2) \\ \vdots \\ X(k_{N_A}) \end{bmatrix} \tag{22}$$

where the measurement matix is defined by

$$\mathbf{A}_K = \begin{bmatrix} e^{-j\frac{2\pi}{N}(n_1k_1+\phi_1)} & \dots & e^{-j\frac{2\pi}{N}(n_Kk_1+\phi_1)} \\ \vdots & \ddots & \vdots \\ e^{-j\frac{2\pi}{N}(n_1k_{N_A}+\phi_1)} & \dots & e^{-j\frac{2\pi}{N}(n_Kk_{N_A}+\phi_{N_A})} \end{bmatrix}$$

with $\phi_i = k_i^2 b_{21} + k_i^3 b_{31}$ for $i = 1, \dots, K$. Starting from the available values $X(k)$, we reconstruct the non-zero values in time $[x_{b_{21},b_{31}}(n_1), x_{b_{22},b_{32}}(n_2), \dots, x_{b_{2K},b_{3K}}(n_K)]$ using the iterative OMP procedure, starting with

$$\mathbf{x}_1 = (\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H\mathbf{y}. \tag{23}$$

After the DPFT sample at $n_1$ are reconstructed then the remaining unavailable values $X(k)$ are calculated for the first component. This component is removed from the original measurements. The procedure is repeated for the second component. After the parameters of the second component are found as $(\beta_{22}, \beta_{32}) = (b_{22}, b_{32})$, both the first and second component are reconstructed using both components

$$(\beta_{21}, \beta_{31}) = (b_{21}, b_{31}), \text{ and } (\beta_{22}, \beta_{32}) = (b_{22}, b_{32}).$$

After the first two values are reconstructed, the procedure is continued for all $n_i$. For the case when the DPFT values are not on the grid, we may use few instances around $n_i$ in the reconstruction. The stopping criterion can be the energy of the remaining signal after the reconstructed components are removed.

## V. Example

We consider a monocomponent signal of the form (1) transmitted through a dispersive channel. The channel is $D = 20$ m deep, with a distance of $r = 2000$ m between the transmitter and the receiver. The sampling frequency is $f_s = 1000$ Hz with frequencies of the modes being randomly positioned between 250 Hz $< f <$ 500 Hz. The transfer function is of the form (2) with $M = 5$ modes and the attenuation rate $A_m = (6-m)W(f)$ where $W(f)$ is the DFT of a Hanning window. The received signal without noise is shown in Fig. 1 (top subplots).

It is assumed that the received signal has disturbances in frequency domain in the form of high-impulses in 10% of the spectrum. This will affect the time domain by introducing new sinusoids. The noisy received signal is illustrated in Fig. 1 (middle subplots).

Firstly, we remove the components which are affected by the noise using notch filters. The noisy spectral samples are considered as unavailable. The frequency and time domains of the received signal after filtering are shown in Fig. 1 (bottom subplots). After that, the parameters are found using a third order DPFT. The reconstructed signal in time domain is shown in Fig. 2 (left). The individual components found by the procedure are shown in Fig. 2 (right). The dual S-method representation as an improved version of short-time Fourier transform [13], [15] is used for displaying time-frequency content of the individual components and reconstructed signal. The S-method of the five decomposed and reconstructed modes is shown in first five subplots of Fig. 3. Sum of the normalized five modes is shown Fig. 3 (bottom right subplot).

## VI. Conclusions

Reconstruction of acoustic signals sparse in the dual polynomial Fourier transform is considered. The signal is considered to be transmitted in a dispersive underwater channel environment. The noisy received signal is reconstructed and decomposed using DPFT and compressive sensing methods.

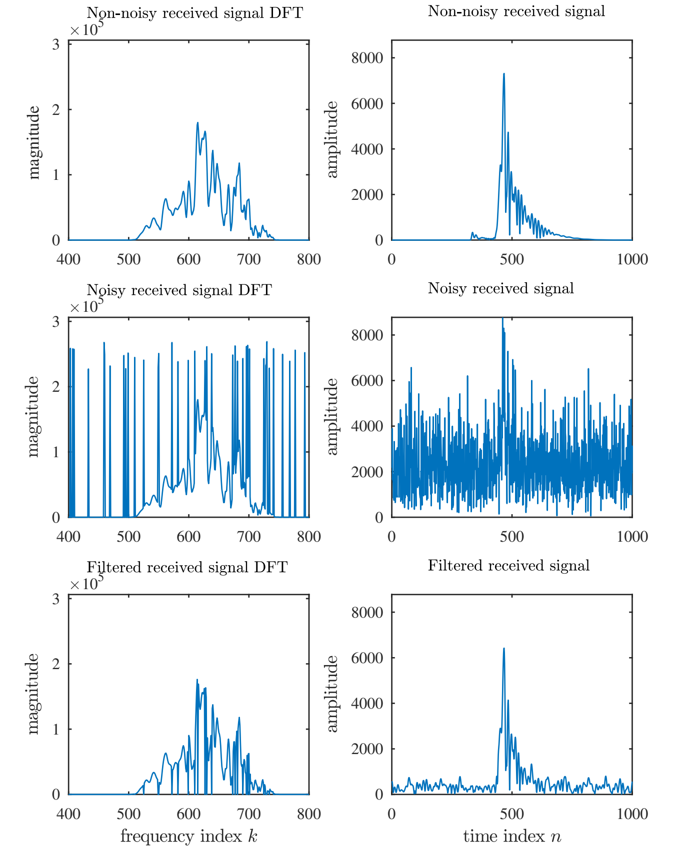


Fig. 1. DFT of received signal (left) and the received signal in time domain (right): without noise (top); with noise (middle) and filtered (bottom)

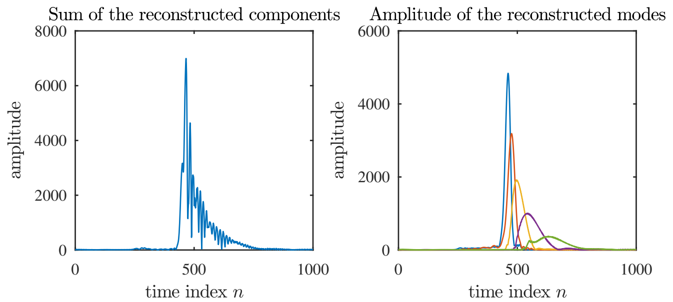


Fig. 2. Reconstructed signal: sum of the modes (left); individual reconstructed modes (right)

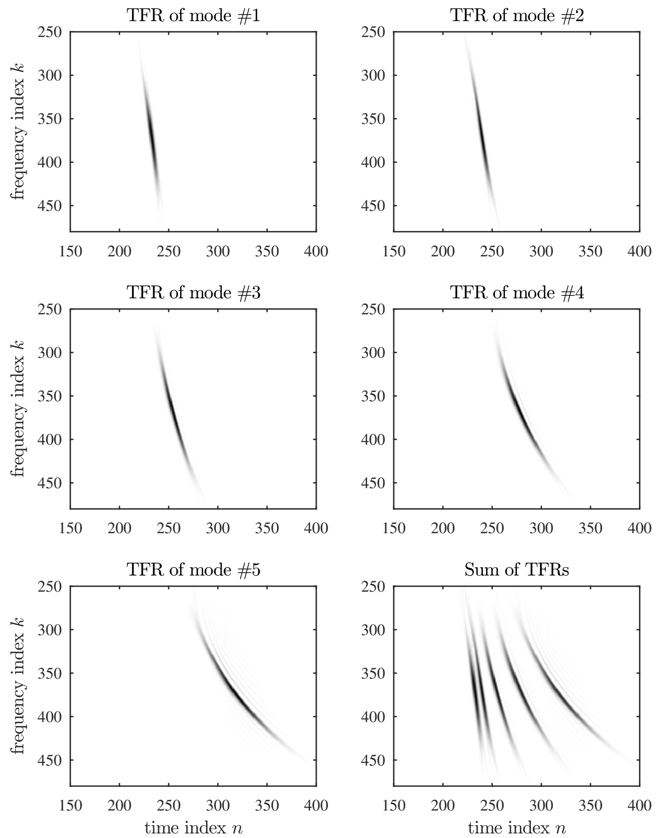


Fig. 3. Decomposition of mode functions in the dual S-method representation domain and sum of the decomposed (normalized) five modes (bottom right)

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