Analysis of off-grid effects in wideband sonar images using compressive sensing

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***Abstract*—In this paper an analysis of sparse wideband sonar images, obtained using compressive sensing reconstruction methods, for generally positioned off-grid targets, is presented. An exact relation is derived for the expected squared error in the resulting sonar image reconstructed from a reduced set of measurements, assuming the sparsity constraint. The error depends on the number of available data, as compared to the complete set of data, and the assumed sparsity. Since the signal is not on the grid, it looses the property of sparsity in the transformation domain. The effects of random sampling and noise will be illustrated and checked on examples as well.**

***Index Terms*—Sonar, Imaging, Compressive sensing, Sparsity**

I. Introduction

Sparse signals are signals that have a small number of nonzero samples in one of its representation domains in comparison to their total length. If a signal is considered as sparse, it can be reconstructed with a reduced set of measurements, which are represented as linear combinations of the sparsity domain coefficients. The reduced set of samples occurs for different reasons. Some of them are to reduce the signal acquisition time, equipment load, energy consumption or required memory. Sometimes the physical unavailability of some measurements or mathematical formulation of the problem results in a reduced set. Also, in many applications, strong disturbances can significantly corrupt the signal and its samples so that they should be omitted from the analysis. Regardless of the reasons why the set of measurements is smaller, under certain reasonable conditions, missing measurements can be reconstructed using the theory of compressive sensing (CS) and sparse signal processing [1]–[3]. Since the introduction of CS, many methods and algorithms for the reconstruction and analysis were developed. Also, CS has a wide range of applications, such as in multimedia, telecommunications, remote sensing, etc.

The usage of CS analysis in radar and sonar signal processing is a topic which was discussed in many ways, with radar imaging being one of them. Radar and sonar imaging systems are used in many everyday applications. Focusing on the sparsity constraint of concerned signals, radar and sonar systems can obtain high quality images with small amount of data, using compressive sensing techniques. One of the famous methods for reducing the number of measurements in radars assumes that the signals which are used in the process of imaging are narrowband [4]. According to the Doppler frequency shifts and delays, point targets can be processed in the range-Doppler plane.

Although the radar and sonar system principles are similar, they have differences which make changes in their configuration. Since sonar systems work under water, the sound velocity is much slower than the velocity of radar waves, which makes the Doppler frequency shifts spreading over larger spectrum. Wideband signal model is usually assumed, making it difficult to transfer traditional radar CS imaging methods directly to the sonar field. Besides frequency variation discussed in the radar imaging with narrowband signals, the scaling effect should always be taken into consideration, particularly for the Doppler effect part. In [5], wideband sonar imaging using compressive sensing was examined. A sampling scheme based on the scaling and frequency shifts was proposed, together with a modified ℓ_1-norm minimization algorithm for sonar imaging. The disadvantage is that it works with the assumption that sparse targets lie on the grid in the range-Doppler plane.

In practice, the targets are not on the grid. This fact increases the sparsity and, in theory, makes the signal nonsparse even in the case of a small number of targets [9]–[12]. In this paper, we will analyze wideband sonar imaging when signal frequencies are not on the grid (when they are only approximately sparse). An exact relation for the expected squared error will be derived, in the resulting sonar image that is reconstructed from a reduced set of measurements, assuming the sparsity constraint. In the literature, this kind of reconstruction error is described by appropriate error bound relations [8]. The error depends on the number of available data, as compared to the complete set of data, and the assumed sparsity. The effects of random sampling and noise will be illustrated and checked on examples as well.

II. Wideband Sonar Image Modelling

Consider a monostatic sonar platform model. Assume that it transmits a signal of the Alltop sequence form [4]

$$x(n) = \frac{1}{\sqrt{N}} e^{j2\pi\frac{n^3}{N}} \qquad (1)$$

with $n = 1, 2, \dots, N$.

The transmitted signal is modulated on a carrier frequency f_c. In the continuous time domain, the modulated transmitted signal is of the form [5]

$$x(t) = \frac{1}{\sqrt{N}} \exp\left(j2\pi \Big\lceil \frac{t}{\Delta} \Big\rceil^3 / N\right) \exp\left(j2\pi f_c t\right) \quad (2)$$

where Δ is the code width of the sequence, $0 \le t < N\Delta$, and f_c is the carrier frequency.

The echo signal from one point target, whose velocity is v, is delayed for τ and scaled in frequency due to the Doppler effect for $(c+v)/(c-v)$. It is presented as

$$r_1(t) = g \frac{1}{\sqrt{N}} \exp\left(2\pi j \Big\lceil \frac{\frac{c+\nu}{c-\nu}(t-\tau)}{\Delta} \Big\rceil^3 / N\right) \times \exp\left(j2\pi f_c \frac{c+\nu}{c-\nu}(t-\tau)\right) \quad (3)$$

where g is a complex scattering coefficient and c is the speed of sound.

For a K scattering points, the received discrete echo signal can be written as a sum of K echoes (3) [4]–[7]

$$r(n) = \sum_{i=1}^{K} g_{k_i} \frac{1}{\sqrt{N}} \exp\left(2\pi j (n - d_{k_i})^3 / N\right) \exp\left(j\omega_{k_i} n\right)$$

where d_k corresponds to the range (time delay) and ω_k to the cross-range (Doppler shift). Assume that the range and cross-range coordinates are on the grid and may assume one of the values from the set

$$(d_p, \omega_q) \in \{d_1, d_2, \dots d_N\} \times \{\omega_1, \omega_2, \dots \omega_N\},$$

where

$$d_k \in \{d_1, d_2, \dots d_N\}$$
$$\omega_k \in \{\omega_1, \omega_2, \dots \omega_N\}. \quad (4)$$

Note that there are N^2 possible positions in total.

In this case, the received echo can be written as

$$r(n) = \sum_{i=1}^{K} g_{k_i} \phi_{k_i}(n) \quad (5)$$

where $\phi_k(n)$ are the basis functions defined by

$$\phi_k(n) = \frac{1}{\sqrt{N}} \exp\left(2\pi j (n - d_k)^3 / N\right) \exp\left(j\omega_k n\right).$$

The basis function for a given pair $(d_p, \omega_q) = (p-1, \frac{2\pi(q-1)}{N})$, corresponding to the scatterer k, can be written as

$$\phi_{p,q}(n) = \frac{1}{\sqrt{N}} \exp\left(2\pi j (n - (p-1))^3 / N\right) \times \exp\left(j2\pi (q-1) n / N\right) \quad (6)$$

and

$$r(n) = \sum_{i=1}^{K} g_{k_i} \phi_{p_{k_i}, q_{k_i}}(n).$$

III. Compressive sensing background

The samples of received signal (5) can be written in a matrix from as

$$\mathbf{r} = \mathbf{\Phi} \mathbf{g}, \quad (7)$$

where $\mathbf{r} = [r(0), r(1), \dots, r(N^2-1)]^T$ is the column vector of the echo signal, $\mathbf{\Phi}$ is the matrix with basis functions and $\mathbf{g} = [g(0), g(1), \dots, g(N^2-1)]^T$ is the column vector of the scattering coefficients $g(k) = g_k$.

Note that the vector $\mathbf{g}$ of scattering coefficients can be sparse since we may assume that the echo is produced by only a few of the possible reflecting points. The number of reflecting points K is assumed to be much smaller than the total number of possible scattering points $N \times N$.

Since the sparsity of the vector $\mathbf{g}$ may be assumed then its values can be reconstructed from a reduced set of observations. The reduced set of received signal samples is

$$\mathbf{y} = [r(n_1), r(n_2), \dots, r(n_M)]^T. \quad (8)$$

The measurements equation is then

$$\mathbf{y} = \mathbf{A} \mathbf{g} \quad (9)$$

where the measurement matrix $\mathbf{A}$ is obtained from the full matrix $\mathbf{\Phi}$ by keeping the rows corresponding to the available samples at $n_i \in \mathbb{M} = \{n_1, n_2, \dots, n_M\}$. The elements of matrix $\mathbf{A}$ are

$$a_{k,l} = \frac{1}{\sqrt{N}} \exp\left(2\pi j (n_l - d_p)^3 / N\right) \exp\left(j\omega_q n_l\right)$$

where d_p and ω_q correspond to the vector rearranged coefficients for a given scattering position k.

A general compressive sensing formulation is

$$\min \|\mathbf{g}\|_0 \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{g}. \quad (10)$$

That is, we try to reconstruct the sparsiest possible reflection coefficients vector $\mathbf{g}$, by minimizing the number of its nonzero elements $\|\mathbf{g}\|_0$ subject to the available samples $\mathbf{y}$. The reconstruction procedure used in this paper is presented next.

IV. Reconstruction Procedure

For the reconstruction of sparse $\mathbf{g}$, we will use its initial estimate as a projection of the available measurements/samples of the received echo on the measurement matrix. The initial estimate is calculated by using only the available measurements

$$\mathbf{g}_0 = \mathbf{A}^H \mathbf{y}. \quad (11)$$

or

$$g_0(k) = \sum_{n_i \in \mathbb{M}} r(n_i) a^*_{k,n_i}. \quad (12)$$

If we replace the echo signal we get

$$g_0(k) = \sum_{n_i \in \mathbb{M}} \sum_{i=1}^{K} g_{k_i} \phi_{p_{k_i}, q_{k_i}}(n_i) a^*_{k,n_i}. \quad (13)$$

Note that, with a random set of available samples, the initial estimate is a random variable. This relation may be used to derive the coherence index based reconstruction relation for the considered signal (see Appendix).

The mean and the variance of $\mathbf{g}_0$ are calculated using the randomly sampled available signal measurements [10]. For $K = 1$ the initial estimate is

$$g_0(k) = \sum_{n_i \in \mathbb{M}} g_{k_1} \phi_{p_{k_1}, q_{k_1}}(n_i) a^*_{k,n_i}$$
$$= \sum_{n_i \in \mathbb{M}} g_{k_1} \frac{1}{\sqrt{N}} e^{2\pi j (n_i - (p_{k_1}-1))^3/N} e^{j2\pi(q_{k_1}-1)n_i/N}$$
$$\times \frac{1}{\sqrt{N}} e^{-2\pi j (n_i-(p-1))^3/N} e^{-j2\pi(q-1)n_i/N}.$$

Its mean for $k = k_1$ is $E\{g_0(k)\} = g_{k_1}M/N$. Since $\mathbb{M}$ is a random set, for $k \neq k_1$ the initial estimate behaves as random variable with zero-mean and variance $\text{var}\{g_0(k)\} = |g_{k_1}|^2 M/N^2$. Now the results can easily be generalized for any sparsity K

$$E\{g_0(k)\} = \sum_{i=1}^{K} \frac{M}{N} g_{k_i} \delta(k - k_i) \quad (14)$$

$$\text{var}\{g_0(k)\} = \frac{M}{N^2} \sum_{i=1}^{K} |g_{k_i}|^2 \left(1 - \delta(k-k_i)\right), \quad (15)$$

where $\delta(k) = 1$ only for $k = 0$ and $\delta(k) = 0$, elsewhere.

A. Reconstruction Algorithm

A variant of the orthogonal matching pursuit (OMP) algorithm will be used for the reconstruction. The signal is reconstructed by estimating the positions of the nonzero components and calculating the unknown signal amplitudes g_k based on the known $y(n_i)$ [13]. In the first step, the position of the largest component is found as

$$k_1 = \arg\max\{\mathbf{g}_0\}.$$

and the first estimate of $\mathbf{g}$ with sparsity $K = 1$ is

$$\mathbf{g}_1 = (\mathbf{A}_1^H \mathbf{A}_1)^{-1} \mathbf{A}_1^H \mathbf{y}.$$

The matrix $\mathbf{A}_1$ is partial matrix from matrix $\mathbf{A}$ keeping only the column corresponding to the found position k_1.

The received echo signal $\mathbf{r}_1 = \mathbf{\Phi}\mathbf{g}_1$ is reconstructed and subtracted from the original signal at the available positions $n_i \in \{n_1, n_2, \ldots, n_M\}$. Then, the new initial estimate of $\mathbf{g}$ is calculated using the difference of $\mathbf{y}$ and $\mathbf{r}_1$ at n_i. The maximum of the initial estimate, with this new measurement vector, is at k_2. A new set of positions of nonzero coefficients in $\mathbf{g}$ is formed as $\mathbb{K} = \{k_1, k_2\}$ and a new matrix $\mathbf{A}_2$ is formed from $\mathbf{A}$ using columns $\{k_1, k_2\}$. The new estimate $\mathbf{g}_2$ and the signal $\mathbf{r}_2 = \mathbf{\Phi}\mathbf{g}_2$ are calculated. The procedure is repeated K times (i.e. the assumed sparsity), for positions $k \in \mathbb{K} = \{k_1, k_2, \ldots, k_K\}$. The method can be explained by the pseudo code

$$\mathbf{K} = \emptyset, \quad \mathbf{y}_r = \mathbf{y}$$
for $i = 1 : K$
$$\mathbf{g}_0 = \mathbf{A}^H \mathbf{y}_r$$
$$k = \arg\{\max_k |\mathbf{g}_0|\}$$
$$\mathbf{K} = \{\mathbf{K}, k\}$$
$$\mathbf{A}_K = \mathbf{A}(K, :)$$
$$\mathbf{g}_K(n) = (\mathbf{A}_K^T \mathbf{A}_K)^{-1} \mathbf{A}_K^T \mathbf{y}$$
$$g_K(k) = g_K(k), \quad k \in \mathbf{K}$$
$$g_K(k) = 0, \quad k \notin \mathbf{K}$$
$$\mathbf{s}_r = \mathbf{\Phi}\mathbf{g}_K$$
$$\mathbf{y}_r = \mathbf{y} - \mathbf{s}_r, \quad \text{for} \quad n \in \mathbf{N_A}$$
end
$$\mathbf{g}_R = \mathbf{g}_K.$$

V. Off-grid Error Analysis

For a sparse reflection coefficient vector $\mathbf{g}$, all K reflecting points should be at the assumed grid in the range and cross-range domain d_k and ω_k. A real scenario is that the reflecting points are off-grid. Then even one reflecting point causes nonzero values at all grid points in the range and cross-range domain (leakage effect in standard spectral analysis). Since the reflecting points are not on the assumed grid, the reflection coefficient vector $\mathbf{g}$ looses the property of strict sparsity [6], [7], [10]. Loosing the sparsity property, the signal becomes nonsparse in this domain.

From the compressive sensing theory, we know that a signal is K-sparse if it has K nonzero coefficients in a transformation domain (in this case reflection coefficients domain) at $k \in \mathbb{K}$, and other coefficients are equal to zero. A signal is considred as nonsparse if there exist nonzero coefficients not only at $k \in \mathbb{K}$, but also at the positions $k \notin \mathbb{K}$ [11].

Let consider a reflected echo signal $\mathbf{r}$ that is not strictly sparse in the coefficients $\mathbf{g}$ domain. In order to use the theory of compressive sensing, the sparsity assumption has to be made. The error which is produced by the reconstruction of nonsparse signal with a sparsity constraint is calculated in this paper. We assume that the CS conditions for the reconstruction are satisfied with assumed sparsity K and the number of available samples M. Then we can detect and reconstruct K components of vector $\mathbf{g}$ using the OMP algorithm and available samples of received echo signal $\mathbf{r}$ given by $\mathbf{y}$.

The reconstructed coefficients vector has K (nonzero) reconstructed components, meaning that $N^2 - K$ coefficients remained unreconstructed. Following (15), one unreconstructed coefficients produces noise in the reconstructed components with variance $|g_i|^2 M/N^2$. The noise variance in the reconstructed coefficients will have a scaling factor of $(N/M)^2$, since the signal amplitudes in the initial estimate are proportional to M and the amplitudes are recovered to their original values as if all samples were available, proportional to its size. Therefore, the variance of noise which causes a single reflection coefficients which is not reconstructed to the reconstructed one is

$$|g_i|^2 \frac{M}{N^2}\left(\frac{N}{M}\right)^2 = \frac{1}{M}|g_i|^2. \tag{16}$$

For a signal reflected from K points, the white noise energy in the reconstructed coefficients will be K times larger than the energy (variance) in one reconstructed coefficient. Then, the total noise caused by the unreconstructed coefficients can be written as

$$\|\mathbf{g}_R - \mathbf{g}_K\|_2^2 = \frac{K}{M}\sum_{i=K+1}^{N^2}|g_i|^2. \tag{17}$$

Energy corresponding to the unreconstructed $N-K$ coefficients is obviously

$$\|\mathbf{g} - \mathbf{g}_K\|_2^2 = \sum_{i=K+1}^{N^2}|g_i|^2. \tag{18}$$

Finally, combining (17) and (18), we can conclude that the error in the reconstructed coefficients with respect to the K corresponding coefficients if the original signal were used is

$$\|\mathbf{g}_R - \mathbf{g}_K\|_2^2 = \frac{K}{M}\|\mathbf{g} - \mathbf{g}_K\|_2^2, \tag{19}$$

where:

- $\|\mathbf{g}\|_2^2 = E\{\sum_k |g(k)|^2\}$ is the expected value of squared norm-two
- $\mathbf{g}_K$ is the K-sparse version of $\mathbf{g}$. The elements of vector $\mathbf{g}_K$ are $g_K(k) = g(k)$ for $k \in \mathbb{K}$, and $g_K(k) = 0$ for $k \notin \mathbb{K}$.
- The reconstructed $\mathbf{g}_R = g_R(k)$ is formed in the same way. The coefficients at $k \in \mathbb{K}$ are the results from the reconstruction procedure and $\mathbf{g}_R = 0$ at $k \notin \mathbb{K}$.

Assume now that the available measurements have additive noise

$$\mathbf{y}_n + \boldsymbol{\varepsilon}_n = \mathbf{A}\mathbf{g}. \tag{20}$$

The variance of additive noise ε is σ_ε^2. Noisy measurements will result in a noisy initial estimate $g_0(k)$

$$g_0(k) = \sum_{n_i \in \mathbb{M}} (r(n_i) + \varepsilon(n_i)) a_{k,n_i}^*.$$

Additive noise caused variance in each term is σ_ε^2/N. Then the total initial estimate variance $\sigma_{g_0(k)}^2 = M\sigma_\varepsilon^2/N$. Since the initial estimate is multiplied by N/M in the reconstruction, the noise variance in the reconstructed component is

$$\mathrm{var}\{g_R(k)\} = \frac{M}{N}\sigma_\varepsilon^2\left(\frac{N}{M}\right)^2 = \frac{N}{M}\sigma_\varepsilon^2.$$

The noise is the same in each reconstructed coefficient. Then the total error in K reconstructed coefficients is [10]

$$\|\mathbf{g}_R - \mathbf{g}_K\|_2^2 = K\frac{N}{M}{\sigma_\varepsilon}^2. \tag{21}$$

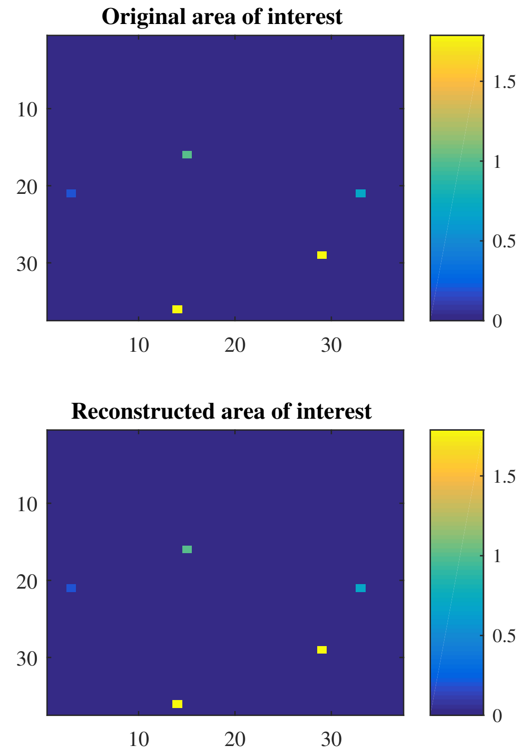

Fig. 1. Sparse case: original image (top); reconstructed image (bottom)

The error will then be calculated as

$$\|\mathbf{g}_R - \mathbf{g}_K\|_2^2 = \frac{K}{M}\|\mathbf{g} - \mathbf{g}_K\|_2^2 + K\frac{N}{M}\sigma_\varepsilon^2. \tag{22}$$

Note that the cases when the reconstruction is ideal, or when original signal is exactly of sparsity K produce no error. If any of the coefficients remains unreconstructed, it will behave as noise in the reconstructed coefficients. Analysis of this error on examples is presented next.

VI. Numerical Results

We will show an example with a strictly sparse signal to show the reconstruction using the OMP algorithm. In Example 2 and 3 we will do the analysis on a nonsparse signal without and with noise, using various sparsity levels and number of available samples.

Example 1: We consider a wideband sonar image of size $N \times N = 37 \times 37$. The original image is shown in Fig. 1 (top). The number of available samples is $M = \lceil 2.5N \rceil = 93$ (a ceil integer close to $2.5N$ as in [5]). The measurement matrix $\mathbf{A}$ is of size $M \times N^2$. We consider that the sparsity level is $K = 5$. The signal is reconstructed using the OMP algorithm. The reconstructed image is shown in Fig. 1 (bottom). Since the signal is exactly sparse, the reconstruction produces no error.

Example 2: Assume a signal which has $K = 5$ targets and another $N^2 - K$ components which are not targets. The

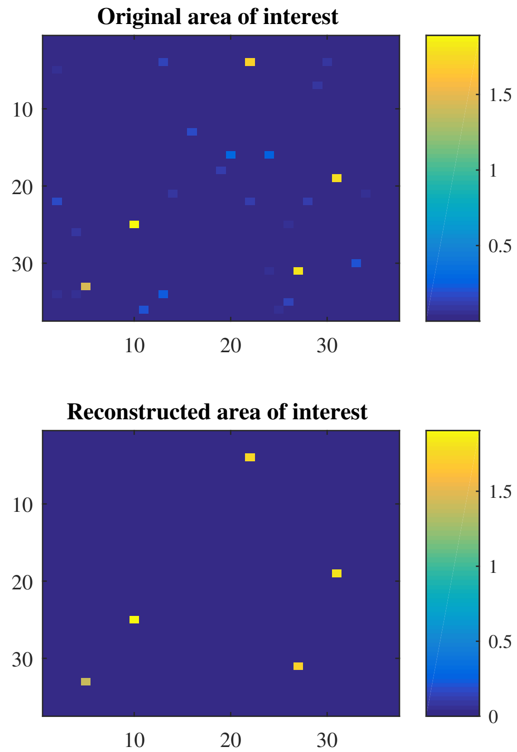

Fig. 2. Nonsparse case: original image (top); reconstructed image (bottom)

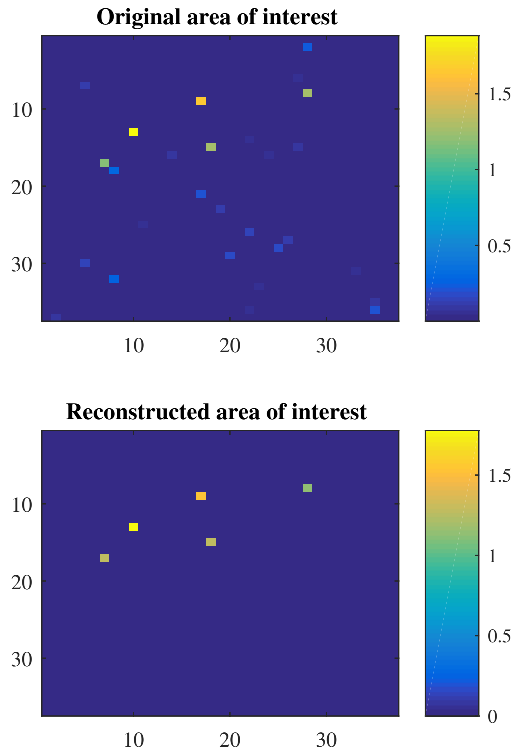

Fig. 3. Noisy nonsparse case: original image (top); reconstructed image (bottom)

nonsparse signal is formed in the following way

$$g(k) = \begin{cases} 1+\nu_k, & \text{for } k \in \mathbb{K} \\ 0.5\exp(-k/K/2), & \text{for } k \notin \mathbb{K} \end{cases}, \tag{23}$$

where $0 \le \nu_k < 1$, are random numbers.

The image is shown in Fig. 2 (top). The signal is reconstructed with the presented iterative OMP and the assumption that it is strictly K-sparse. The reconstructed image is shown in Fig. 2 (bottom).

We will compare our results with the results obtained by using the standard statistical calculation

$$E_s = 10\log\left(\|\mathbf{g}-\mathbf{g}_R\|_2^2\right). \tag{24}$$

The total theoretical error, derived in Section IV, is

$$E_t = 10\log\left((\frac{K}{M}+1)\,\|\mathbf{g}-\mathbf{g}_K\|_2^2\right). \tag{25}$$

We calculated the error using various sparsity levels $K = 1,2,3,4,5,6,7$ in 100 realizations with two numbers of the available samples $M = 2.5N$ and $M = 959$ ($M = \lceil 0.7N^2 \rceil$). The results are shown in Table I.

Example 3: Assume a nonsparse signal of the form (23) with additive noise variance $\sigma_\varepsilon^2 = 1/N$. The signal is reconstructed under the assumption that it is $K = 5$ sparse. The original and reconstructed images are shown in Fig. 3.

The statistical and theoretical error, presented in Table II for various assumed sparsity levels, are calculated as

$$E_s = 10\log\left(\|\mathbf{g}-\mathbf{g}_R\|_2^2\right), \tag{26}$$

$$E_t = 10\log\left((\frac{K}{M}+1)\,\|\mathbf{g}-\mathbf{g}_K\|_2^2 + K\frac{N}{M}\sigma_\varepsilon^2\right). \tag{27}$$

VII. Conclusions

The analysis of off-grid wideband sonar images, obtained using compressive sensing reconstruction method, is presented. The off-grid effect makes the signal nonsparse in the transformation domain. The resulting sonar image is reconstructed from a reduced set of measurements, assuming the sparsity constraint. An exact relation for the expected squared error in the reconstruction is presented. The effects of random sampling and noise are checked on examples and compared to the statistical error calculation.

VIII. Appendix

The initial estimate

$$g_0(k) = \sum_{n_i \in \mathbb{M}} r(n_i)a^*_{k,n_i}$$

for a K sparse signal

$$r(n) = \sum_{k=1}^{K} g_k \phi_{p_k,q_k}(n)$$

can be written as

$$g_0(k) = \sum_{n_i \in \mathbb{M}} \sum_{k=1}^{K} g_k \phi_{p_k,q_k}(n_i)a^*_{k,n_i}.$$

TABLE I
ERROR IN NONSPARSE IMAGE WITH $N = 37$, VARIOUS SPARSITY LEVELS AND $M = 2.5N$ AND $M = 0.7N^2$

$M = 2.5N$	$K = 1$	2	3	4	5	6	7
Statistics	-12.6	-8.3	-6.1	-4.5	-3.1	-2.2	-1.2
Theory	-12.7	-8.4	-6.2	-4.7	-3.5	-2.5	-1.9

$M = 0.7N^2$	$K = 1$	2	3	4	5	6	7
Statistics	-12.6	-8.4	-6.2	-4.7	-3.4	-2.6	-1.8
Theory	-12.7	-8.5	-6.3	-4.9	-3.8	-2.9	-2.2

TABLE II
ERROR IN NONSPARSE IMAGE WITH $N = 37$, NOISE $\sigma_\varepsilon^2 = 1/N$, VARIOUS SPARSITY LEVELS AND $M = 2.5N$ AND $M = 0.7N^2$

$M = 2.5N$	$K = 1$	2	3	4	5	6	7
Statistics	-12.1	-7.9	-5.6	-4.0	-2.9	-2.1	-1.0
Theory	-12.2	-8.1	-5.9	-4.4	-3.3	-2.4	-1.6

$M = 0.7N^2$	$K = 1$	2	3	4	5	6	7
Statistics	-12.5	-8.3	-6.2	-4.6	-3.4	-2.5	-1.8
Theory	-12.6	-8.4	-6.3	-4.8	-3.7	-2.9	-2.1

The maximal absolute value of $\sum_{n_i\in\mathbb{M}} \phi_{p_k,q_k}(n_i)a^*_{k,n_i}$ is defined as the coherence index

$$\mu = \max|\sum_{n_i\in\mathbb{M}} \phi_{p_k,q_k}(n_i)a^*_{k,n_i}|$$

Assume that the maximal reflection coefficient g_k is normalized $\max|g_k| = 1$. The strongest influence of other $K-1$ coefficients to this coefficient will be if they are almost the same and equal to 1. The reconstruction of the strongest coefficient is always possible if its value 1 being maximally reduced with other coefficients to $1-(K-1)\mu$ is greater than the maximal possible value of all K coefficients at a noise position being $K\mu$. It means that the reconstruction of strongest component is guaranteed if

$$1-(K-1)\mu > K\mu$$

or

$$K < \frac{1}{2}(1+\frac{1}{\mu}).$$

This the famous coherence based reconstruction relation. It is extremely pessimistic, as can easily be concluded from this simple derivation, since we considered the worst case several times in its derivation. For the Alltop sequence the coherence index can be approximated by $\mu = \frac{1}{\sqrt{N}}$ for $M = N^2$ and $K < \frac{1}{2}(\sqrt{N}+1)$ follows as a very strict bound.

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