Complex-Valued Binary Compressive Sensing

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***Abstract* — One-bit (or binary) compressive sensing (CS) is a relatively new idea in the theory of sparse signal reconstruction. It is based on using only the sign of the available measurements for the signal recovery. In this paper, we analyze the one-bit CS concepts on complex-valued random Gaussian measurement matrices. The signal is reconstructed using an iterative hard thresholding algorithm, modified for the complex-valued binary measurements. The considered CS approach is particularly suitable for hardware realizations. The reconstruction performance is validated numerically, and compared with the traditional CS reconstruction based on quantized digital measurements.**

***Keywords* — compressive sensing, complex, binary, one-bit, reconstruction, sparse signal processing**

I. Introduction

Compressive sensing (CS) is a still growing field dealing with sparse signals [1]–[7]. Comparing to the total amount of samples in a signal, a sparse signal has a small number of nonzero coefficients in one of its representation domains. The theory of CS states that these signals can be reconstructed from a reduced set of observations. The advantages of CS are in the signal transmission and storage efficiency, which is crucial in big data setups. Unavailable signal samples can occur as an intentional sampling strategy or constraints of physical systems. In certain applications, some samples could intentionally be thrown away due to a high level of corruption, after application of some particular signal processing techniques. Illustrative example is the application of L-statistics to the removal of highly corrupted noisy samples [4]. Since the introduction of the theory, many reconstruction theorems and algorithms were developed [8]–[13].

In real applications, many signals are sparse in a certain domain. This makes the CS applicable in various fields of signal processing, including the biomedicine, telecommunications, media, etc [12]. Ideally, the measurements used for the reconstruction should be taken accurately, assuming a very large number of bits in their digital form. However, this could be extremely demanding and expensive for hardware implementation [14]. That is the reason why, in practice, the measurements are quantized to a certain level, using a limited number of bites. In this paper, we consider the most extreme case – using only one bit to represent the measurements. This kind of measurements is very robust, memory efficient, and simple for sensors design. In particular, we analyze complex-valued signals, which appear in many signal processing applications.

In previous work [14]–[17], one-bit measurements are initially treated as sign constraints, as opposed to the values to be matched in the mean squared sense during the reconstruction process. As signs of measurements do not provide amplitude information of the signal, the signal can be recovered up to a constant scalar factor only. In the one-bit CS framework, by imposing a unity energy constraint on the reconstructed signal, the ambiguity is resolved and the signal reconstruction is performed [14].

Quantization to one-bit measurements is particularly suitable for hardware systems. The quantizer for the algorithm presented in this paper takes the form of a comparator to zero, which is an inexpensive and fast hardware device. Moreover, one-bit quantizers do not suffer from dynamic range issues [14]–[17]. In [18], a complex form has been used in direction-of-arrival estimation with a partial Fourier transform matrix as a measurement matrix. In this paper, a general Gaussian distributed measurement complex matrix is considered. The results with a binary complex algorithm are compared with a digital algorithm using B bits with respect to number of iterations and accuracy, extending the real case analysis from [15].

The paper is organized as follows. In Section II, the traditional compressive sensing is briefly explained. The idea of binary compressive sensing and the complex-valued extension are shown in Section III. In Section IV, the reconstruction algorithm is presented. Results and comparison are shown in Section V and the conclusions are presented in Section VI.

II. Traditional Compressive Sensing

Consider a discrete signal $x(n)$ of length N, with $1 \leq n \leq N$. Its transform domain representation is denoted by $X(k)$. The signal and the corresponding transform are related as

$$x(n)=\sum_{k=0}^{N-1}X(k)\psi_k(n), \qquad X(k)=\sum_{n=0}^{N-1}x(n)\phi_n(k),$$

or in vector form

$$\mathbf{x}=\mathbf{\Psi}\mathbf{X}, \qquad \mathbf{X}=\mathbf{\Phi}\mathbf{x}. \tag{1}$$

The matrices $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are the direct and inverse transformation matrices, respectively. We assume that the signal is K-sparse in the transformation domain, meaning that it has $K \ll N$ nonzero coefficients in that domain.

Following the compressive sensing framework, we can use only $M < N$ samples to reconstruct this signal. The signal with the available M observations will be denoted by $y(m)$, $m=1,2,\dots,M$, written as

$$\mathbf{y}=y(m)=[x(n_1),x(n_2),\dots,x(n_M)]^T. \tag{2}$$

The general goal of compressive sensing is to minimize the sparsity of $\mathbf{X}$ using only the available observations $\mathbf{y}$. This reconstruction problem can be formulated as an optimization problem

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$$\min \ \|\mathbf{X}\|_0 \ \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{X}, \tag{3}$$

where $\mathbf{y}$ are the samples/measurements and $\mathbf{A}$ is the corresponding $M \times N$ measurement matrix. Its rows correspond to the positions of the available samples. The measurement matrix consist of i.i.d. standard normal distribution coefficients. In that case, there is no a direct/inverse transform framework for the measurements and measurement matrix.

The ℓ_0-norm used in (3) counts the number of nonzero coefficients in $\mathbf{X}$. However, this norm is not convex and is very sensitive (not applicable) to the noisy signal cases. This the reason why, in practice and theory, more robust norms are used to measure the sparsity. The most frequently used one is the ℓ_1-norm

$$\min \ \|\mathbf{X}\|_1 \ \text{ subject to } \mathbf{y} = \mathbf{A}\mathbf{X}. \tag{4}$$

To solve this problem, many algorithms were developed in recent years.

III. Binary Compressive Sensing

In the CS theory, the number of measurement bits is not commonly considered. Bit-limiting the measurements could significantly affect the reconstruction performance of the standard CS approaches. The measurement quantization is particularly important in the hardware implementation context. One-bit measurements are the most extreme case, promising simple, comparator-based hardware devices. The one bit used represents the sign of the sample

$$\mathbf{y} = \text{sign}\{\mathbf{A}\mathbf{X}\} \tag{5}$$

which is

$$y(m) = \begin{cases} 1, & \text{for the elelemnts of } \mathbf{A}\mathbf{X} > 0 \\ -1, & \text{for the elelemnts of } \mathbf{A}\mathbf{X} \leq 0. \end{cases} \tag{6}$$

Then, the CS reconstruction goal can be reformulated as

$$\min \ \|\mathbf{X}\|_1 \ \text{ subject to } \mathbf{y} = \text{sign}\{\mathbf{A}\mathbf{X}\}. \tag{7}$$

Using this method, we try to reconstruct the coefficients $X(k)$ by ignoring the amplitude of the available measurements. Consequently, this method needs more measurements, i.e. $M > N$, so the traditional idea of CS is, in that sense, destructed. However, in the case when the quantization accuracy is more important than the number of measurements, we can afford having a bigger number of samples [14]. The requirement for storage is also significantly reduced for sign measurements since the total number of bits is still low.

A. Complex-valued extension

Further, we assume a complex-valued signal $x(n)$. Then the measurements $\mathbf{y} = \mathbf{A}\mathbf{X}$ are also complex. We assume the measurement matrix $\mathbf{A}$ to consist of random, complex-valued coefficients, with identical, normal zero-mean distribution of real and imaginary parts, with a unity variance

$$\psi_k(n_m) \sim \mathcal{N}(0,1) + j\mathcal{N}(0,1) \tag{8}$$

for $k = 0, 1, \ldots, N-1$ and $m = 1, 2, \ldots, M$. In that case, the measurements are formed as

$$\mathbf{y} = \text{sign}\{\Re\{\mathbf{A}\mathbf{X}\}\} + j\,\text{sign}\{\Im\{\mathbf{A}\mathbf{X}\}\}, \tag{9}$$

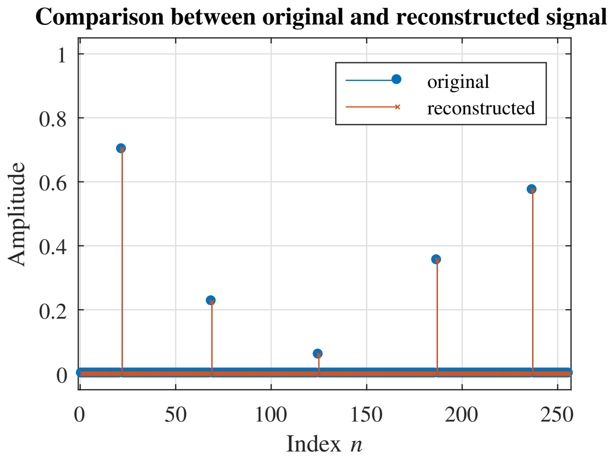

Fig. 1. Reconstruction of a signal using the BIHT algorithm, based on the measurements with one-bit real and imaginary parts. The signal parameters are $N = 256$, $M = 3 \times 256$ and $K = 5$.

where both real and imaginary parts are one-bit quantized.

IV. Binary Iterative Hard Thresholding Algorithm

For the reconstruction, we will adapt a method from the family of greedy algorithms, the Binary Iterative Hard Thresholding (BIHT) algorithm presented in [15], [16] for complex-valued signals. It is an extended version of the Iterative Hard Thresholding (IHT) algorithm [10]. It is convenient for the one-bit compressive sensing framework.

For the binary case, the CS is then defined as solving

$$\begin{gathered} \text{argmin}\, ||\mathbf{y} - \text{sign}(\Re\{\mathbf{A}\mathbf{X}) - j\,\text{sign}(\Im\{\mathbf{A}\mathbf{X}\})||_2^2 \\ \text{s.t. } ||\mathbf{X}||_0 = K. \end{gathered} \tag{10}$$

The BIHT algorithm can be divided in three steps.

1. The first step solves the gradient descent in the least square sense. Given that the initial estimate is $\mathbf{X}_i = \mathbf{0}$ when $i = 0$, the value of the signal is updated at the iteration i as

$$\mathbf{a}_{i+1} = \mathbf{X}_i + \frac{\tau}{2}\mathbf{A}^T\left[\mathbf{y} - \text{sign}(\Re\{\mathbf{A}\mathbf{X}_\mathbf{i}) - j\,\text{sign}(\Im\{\mathbf{A}\mathbf{X}_\mathbf{i}\})\right] \tag{11}$$

where τ is a constant regulating the gradient descent step.

2. We update the value $\mathbf{X}_i$ by finding the K largest components in $\mathbf{a}_{i+1}$

$$\mathbf{X}_i = \text{T}_K\{\mathbf{a}_{i+1}\}, \tag{12}$$

where $\text{T}_K\{.\}$ is the thresholding operator, taking only the K largest components.

The first two steps are repeated until a desired error or the consistency is achieved (or, in the worst case, if the maximum number of iterations is achieved).

3. After the procedure is stopped and the signal is reconstructed, the final step is to normalize the estimate and project the approximation onto the unit ℓ_2 sphere

$$\mathbf{X}_R = \frac{\mathbf{X}_i}{||\mathbf{X}_i||_2}. \tag{13}$$

The algorithm is numerically evaluated in the next section.

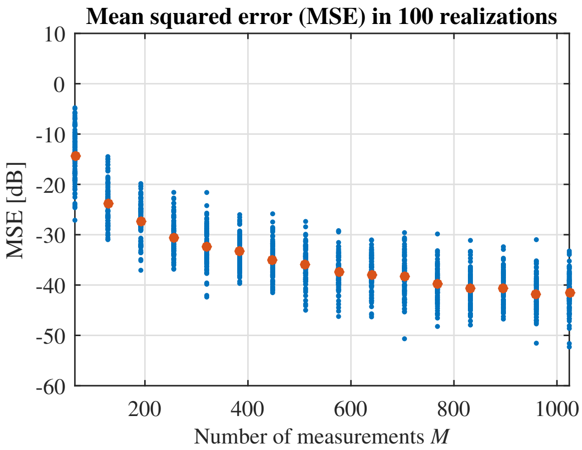

Fig. 2. MSE in the reconstruction of signals with one-bit real and imaginary parts, with length $N = 256$ and sparsity $K = 5$. The MSE is calculated for 100 independent realizations of signals and measurements, for each considered number of measurements M. Blue dots denote mean squared errors from each realization, red dots represent the MSE obtained averaging the results from all realizations.

The binary measurements can be generalized using the B-bit digital form of measurement. For this purpose, it has been assumed that the measurements are stored into B-bit registers,

$$\mathbf{y} = \text{digital}_B\{\mathbf{AX}\}.$$

In this case, the reconstruction is performed using the iterative matching pursuit algorithm.

V. Results

Example 1: Let us consider a signal $x(n)$ of the length $N = 256$. The coefficients of the measurement matrix $\mathbf{A}$ are complex-valued i.i.d. standard normal random variables, that is, $\psi_k(n_m) \sim \mathcal{N}(0,1) + j\mathcal{N}(0,1)$ for $k = 0, 1, \ldots, N-1$ and $m = 1, 2, \ldots, M$. The signal is considered to be sparse with sparsity level $K = 5$. It is reconstructed using the BIHT algorithm based on $M = 3N$ measurements with one-bit real and imaginary parts. The gradient-descent step is $\tau = 0.001$. The original and the reconstructed signals are compared in Fig. 1.

The reconstruction performance of the BIHT algorithm was also checked in a more extensive numerical experiment. For signal with sparsity $K = 5$, the number of measurements M was varied from 64 to 1024, with step 64. For each observed number of measurements M, the mean squared error (MSE) is calculated based on 100 independent random realizations of signal with sparsity $K = 5$, with corresponding random realizations of matrix $\mathbf{A}$. The results are shown in Fig. 2, where blue dots represent squared errors in each observed realization, whereas the red dots show the corresponding MSE values.

Example 2: In this experiment, we observe a signal of length $N = 128$, with sparsity $K = 2$. The original, complex-valued measurements $y(m)$, $m = 1, 2, \ldots, M$ are intentionally brought as inputs of a $B = 8$-bit quantizer, to obtain quantized measurements $\tilde{y}(m)$. The real and imaginary parts are quantized independently, each with 8-bits, to form $\tilde{y}(m) = \lfloor \Re\{y(m)\} \rfloor_B + j\lfloor \Im\{y(m)\} \rfloor_B$, where $\lfloor \cdot \rfloor_B$ denotes the B-bit quantization. These quantized 8-bit measurements are used as the basis of the CS reconstruction using the classical OMP algorithm from [8], [11], [12]. The number of available 8-bit measurements M_{OMP} was varied from 8 to 128, with step 8. The reconstruction MSE was calculated based on 100 independent random realizations of signal with various, randomly positioned non-zero coefficients with random amplitudes, and with corresponding random coefficients in the measurement matrix $\mathbf{A}$, $\psi_k(n_m) \sim \mathcal{N}(0,1) + j\mathcal{N}(0,1)$ for $k = 0, 1, \ldots, N-1$ and $m = 1, 2, \ldots, M_{OMP}$.

The obtained results are shown in Fig. 3 (second row), where the quantization with $B = 8$ bits increases the MSE level, when compared to the reconstruction using accurate complex-valued measurements.

Signals with the same properties as in the OMP case, are considered in the experiment using the BIHT algorithm with $M = B \cdot M_{OMP}$ one-bit measurements (with real and imaginary one-bit parts). The reconstruction MSE is shown in Fig. 3 (first row). The MSE slowly reduces as the number of measurements increases, but the BIHT algorithm does not reach the level of the OMP MSE, for the observed number of measurements, and for the observed quantization level in the case of the OMP algorithm.

To additionally check the influence of the quantization on the OMP reconstruction, we reduce the number of measurement bits to $B = 6$, and perform the same experiment. The results are presented in Fig. 4. They indicate that, in this case, the OMP and BIHT reconstruction performance is similar. With the increase of the number of available one-bit measurements, the MSE of the BIHT reduces proportionally. It is important to note that the main advantage of the BIHT reconstruction is in much simpler hardware implementation, as the measurement procedure requires only the detection of binary values.

VI. Conclusions

In this paper, we analyzed the reconstruction of signals by using the idea of binary measurements in the compressive sensing framework. The binary compressive sensing was applied on complex-valued signals with similar reconstruction results as in the case of using the all-real one-bit version of the approach. We compared the reconstruction results with the classical OMP-based signal recovery from a reduced set of quantized measurements. The measurements quantized to 6 and 8 bits are considered in this context. The traditional CS reconstruction cannot be used on one-bit measurements. The reconstruction obtained by using the one-bit BIHT is comparable with the standard CS recovery based on the quantized measurements, assuming a similar number of the total measurements bits.

Measurement quantization reduces the MSE level in the traditional OMP algorithm. Our further research is oriented towards the quantification of this phenomenon through the derivation of the relation between the MSE level and number of quantization bits.

References

[1] D. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, 2006, pp. 1289–1306.

[2] R. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, no. 4, 2007, pp. 118–121.

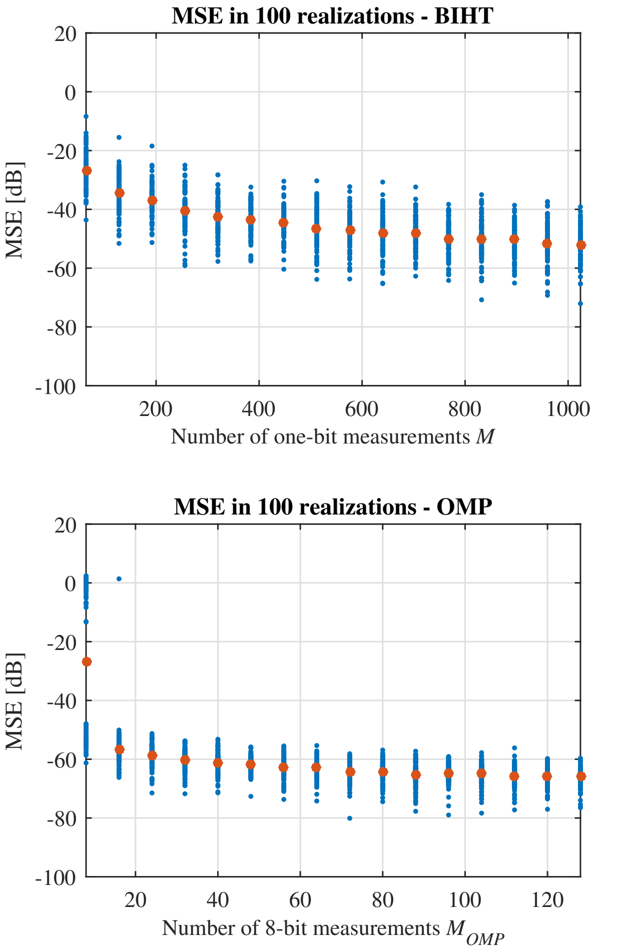

Fig. 3. The mean squared error in the reconstruction of signals with length $N = 128$ and sparsity $K = 2$ using: the BIHT algorithm (first row) and the OMP algorithm (second row). The MSEs are calculated based 100 independent realizations of signals for each considered number of measurements. In the OMP signal case, M_{OMP} random 8-bit quantized measurements are considered (real and imaginary parts are quantized independently). In the case of BIHT, the corresponding number of measurements is $M = 8M_{OMP}$. Blue dots denote mean squared errors in each realization, red dots represent the MSE obtained averaging the results from all realizations.

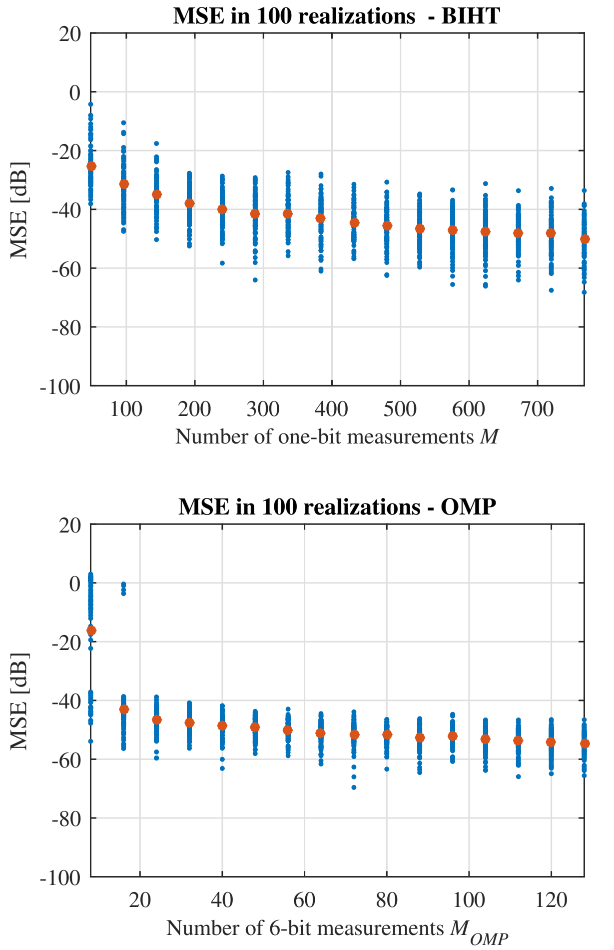

Fig. 4. The mean squared error in the reconstruction of signals with length $N = 128$ and sparsity $K = 2$ using: the BIHT algorithm (first row) and the OMP algorithm (second row). The MSEs are calculated based on 100 independent realizations of signals for each considered number of measurements. In the OMP signal case, M_{OMP} random 6-bit quantized measurements are considered (real and imaginary parts are quantized independently). In the case of BIHT, the corresponding number of measurements is $M = 6M_{OMP}$. Blue dots denote mean squared errors in each realization, red dots represent the MSE obtained averaging the results from all realizations.

[3] E. Candes, J. Romberg and T. Tao. "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on Information Theory*, vol. 52, pp. 489–509, 2006.

[4] L. Stanković, S. Stanković, and M. Amin, "Missing Samples Analysis in Signals for Applications to L-estimation and Compressive Sensing," *Signal Processin*, vol. 94, Jan 2014, pp. 401-408

[5] M. Davenport, M. Duarte, Y. Eldar, G. Kutyniok, "Introduction to compressed sensing," *Chapter in Compressed Sensing: Theory and Applications*, Cambridge University Press, 2012.

[6] L. Stanković, and M. Daković, "On the Uniqueness of the Sparse Signals Reconstruction Based on the Missing Samples Variation Analysis," *Mathematical Problems in Engineering*, vol. 2015, Article ID 629759, 14 pages, 2015. doi:10.1155/2015/629759

[7] S. Stanković, L. Stanković, and I. Orović, "A Relationship between the Robust Statistics Theory and Sparse Compressive Sensed Signals Reconstruction,"*IET Signal Processing*, vol. 8, no. 3, May 2014.

[8] D. Needell, J. A. Tropp, "CoSaMP: Iterative signal recovery from incomplete and inaccurate samples," *Applied and Computational Harmonic Analysis* vol.26, no.3, 2009, pp.301–321.

[9] T. Blumensath, "Sampling and reconstructing signals from a union of linear subspaces," *IEEE Trans. on Inf. Theor.* vol.57, no.7, 2011, pp.4660–4671.

[10] T. Bumensath and M. E. Davies, "Iterative hard thresholding for compressed sensing," *Appl. Comput. Harmon. A.*, vol. 23, no. 3, pp. 265–274, 2009.

[11] L. Stanković, Digital Signal Processing with Selected Topics. CreateSpace Independent Publishing Platform, An Amazon.com Company, November 4, 2015.

[12] L. Stanković, E. Sejdić, S. Stanković, M. Daković, and I. Orović "A Tutorial on Sparse Signal Reconstruction and its Applications in Signal Processing" *Circuits, Systems & Signal Processing*, published online 01. Aug. 2018, DOI 10.1007/s00034-018-0909-2.

[13] L. Stanković, M. Daković, I. Stanković, and S. Vujović, "On the Errors in Randomly Sampled Nonsparse Signals Reconstructed with a Sparsity Assumption,"*IEEE Geoscience and Remote Sensing Lett.*, Vol: 14, Issue: 12, Dec. 2017, pp. 2453 - 2456 , DOI: 10.1109/LGRS.2017.2768664

[14] P. Boufounos, R. Baraniuk, "1-bit Compressive Sensing," *42nd Annual Conference on Information Sciences and Systems*, Princeton, NJ, USA, 2008.

[15] L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, "Robust 1-bit compressive sensing via binary stable embeddings for sparse vectors," *IEEE Transactions of Information Theory*, vol. 59. no. 4, pp. 2082–2102, 2011.

[16] P. North, "One-Bit Compressive Sensing with Partial Support Information," *CMC Senior Theses*, paper 1194, [Available online: http://scholarship.claremont.edu/cmc_theses/1194], 2015.

[17] P. T. Boufounos, "Greedy sparse signal reconstruction from sign measurements," *2009 Conference Record of the 43rd Asilomar Conference on Signals, Systems and Computers*, CA, USA, 2009.

[18] C. Stockle, J. Munir, A. Mezghani and J. A. Nossek, "1-bit direction of arrival estimation based on Compressed Sensing," IEEE 16th International Workshop on Signal Processing Advances in Wireless Communications (SPAWC) 2015, Stockholm, 2015, pp. 246-250.