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Short communication

On the sparsity bound for the existence of a unique solution in compressive sensing by the Gershgorin theorem

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ABSTRACT

Since compressive sensing deals with a signal reconstruction using a reduced set of measurements, the existence of a unique solution is of crucial importance. The most important approach to this problem is based on the restricted isometry property which is computationally unfeasible. The coherence index-based uniqueness criteria are computationally efficient, however, they are pessimistic. An approach to alleviating this problem has been recently introduced by relaxing the coherence index condition for the unique signal reconstruction using the orthogonal matching pursuit approach. This approach can be further relaxed and the sparsity bound improved if we consider only the solution existence rather than its reconstruction. One such improved bound for the sparsity limit is derived in this paper using the Gershgorin disk theorem.

1. Introduction

In compressive sensing we are dealing with a reduced set of signal observations [1–10]. The reduced set of observations can be caused by a desire to compressively acquire signal measurements or by physical unavailability to measure the signal at all possible sampling positions and to get a complete set of samples [4]. In some applications, signal samples may be so heavily corrupted at some arbitrary positions that their omission could be the best approach to their processing, when we are left with a reduced set of signal samples as a basis for signal reconstruction [11–13]. The fundamental condition to fully reconstruct the signal from a reduced set of observations is the signal sparsity in a transformation domain. This type of reconstruction is supported by rigorous mathematical framework [5,15–17]. Applications of compressive sensing methods are numerous, including radar signal processing [18,19], time-frequency analysis [20–22], data hiding [23], wireless communications [24], image processing [13], and graph signal processing [14].

While compressive sensing provides a basis for signal reconstruction, assuming the sparsity in a transformation domain, the uniqueness of the solution is of crucial importance, due to the reduced set of measurements. The most comprehensive uniqueness condition has been defined through the restricted isometry property and is characterized by its computational infeasibility. An alternative approach is based on the coherence index. However, this criterion may be quite pessimistic.

An approach to improve the coherence index-based bound has been proposed in [26] by analyzing the initial estimate and the support uncertainty principle as in Stanković [27], Elad and Bruckstein [28]. The approach presented in [26] guarantees unique reconstruction of a sparse signal using the orthogonal matching pursuit approach. In this paper, a relaxed coherence index condition will be derived for the existence of the unique solution of the compressive sensing problem, using the Gershgorin disk theorem. This result guarantees the existence of a unique solution, but not its reconstruction, meaning that the obtained bound can be relaxed as compared to the one introduced in [26]. The new result for the sparsity bound will be related to the classical one and those proposed in [26]. The theory is illustrated by numerical examples.

2. Review of basic definitions

The basic definitions of compressive sensing will be reviewed first, along with the introduction and explanation of the notation used in the next sections.

2.1. Sparse signal

Consider an N-dimensional signal, $\mathbf{x}$, and one of its linear transforms, $\mathbf{X}$, such that $\mathbf{x} = \mathbf{\Phi}\mathbf{X}$, where $\mathbf{\Phi}$ is an $N \times N$ inverse transformation matrix. The transform elements are denoted by $\mathbf{X} =$

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$[X(0),\ X(1), \ldots,\ X(N-1)]^T$, where T represents the transpose operation. This signal is sparse in the considered transform domain if the number of nonzero elements of $\mathbf{X}$, denoted by K, is much smaller than the signal dimension, N, that is, if the following property holds

$$X(k) = 0 \quad \text{for} \quad k \notin \mathbb{K} = \{k_1, k_2, \ldots, k_K\} \subset \{0, 1, \ldots, N-1\} \tag{1}$$

and $K \ll N$. The number of nonzero elements, K, can be expressed using the ℓ_0-norm of the vector $\mathbf{X}$ or the cardinality of the set $\mathbb{K}$, as $K = \|\mathbf{X}\|_0 = \text{card}\{\mathbb{K}\}$.

2.2. Measurements

The measurements of the sparsity domain elements are defined as their linear combinations

$$y(m) = \sum_{k=0}^{N-1} a_k(m) X(k), \tag{2}$$

where $m = 0, 1, \ldots, M-1$ is the measurement index and $a_k(m)$, $k = 0, 1, \ldots, N-1$, are the weighting coefficients of the mth measurement. The measurement vector is denoted by $\mathbf{y} = [y(0),\ y(1),\ \ldots, y(M-1)]^T$. Within the framework of linear algebra, the measurements can be considered as an undetermined system with $M < N$ equations,

$$\mathbf{y} = \mathbf{A}\mathbf{X}, \tag{3}$$

where $\mathbf{A}$ is the measurement matrix with elements $a_k(m)$. The size of the measurement matrix is $M \times N$.

In some applications, the measurements represent the acquired signal samples, $\mathbf{y} = \mathbf{\Psi}\mathbf{x}$, where $\mathbf{\Psi}$ is an $N \times M$ random permutation matrix. Since $\mathbf{y} = \mathbf{\Psi}\mathbf{x} = \mathbf{\Psi}\mathbf{\Phi}\mathbf{X}$, this case reduces to (3), with $\mathbf{\Psi}\mathbf{\Phi} = \mathbf{A}$.

2.3. Sparsity aware system

The fact that the signal must be sparse in a transformation domain, with $X(k) = 0$ for $k \notin \mathbb{K} = \{k_1, k_2, \ldots, k_K\}$, is not taken into account within the measurement matrix $\mathbf{A}$ since, in general, the positions of the nonzero values of $X(k)$ are unknown and should be determined. If we assume that the nonzero positions are found (or assumed or known in advance), meaning that $X(k) = 0$ for $k \notin \mathbb{K}$, then a system with a reduced number of unknowns, $\mathbf{X}_K = [X(k_1),\ X(k_2), \ldots,\ X(k_K)]^T$, is obtained. This system corresponds to a reduced $M \times K$ measurement matrix $\mathbf{A}_K$. The system of equations then assumes the form

$$\mathbf{y} = \mathbf{A}_K \mathbf{X}_K. \tag{4}$$

Since $K < M$ must hold in compressive sensing, this system is now an overdetermined system of linear equations. The reduced measurement matrix $\mathbf{A}_K$ is formed using the positions $k \in \mathbb{K}$ of nonzero samples of $\mathbf{X}$. The matrix $\mathbf{A}_K$ directly follows from the measurement matrix, $\mathbf{A}$, when the columns corresponding to the zero-valued elements in $\mathbf{X}$ are omitted. The reconstructed vector, $\mathbf{X}_K$, with the determined/assumed/known nonzero positions, is a solution in the least-squares sense, given by

$$\mathbf{X}_K = (\mathbf{A}_K^H \mathbf{A}_K)^{-1} \mathbf{A}_K^H \mathbf{y}, \tag{5}$$

where $\mathbf{A}_K^H$ is a Hermitian transpose of $\mathbf{A}_K$. The condition for this least-squares reconstruction is the invertibility of the matrix $\mathbf{A}_K^H \mathbf{A}_K$. This condition is much weaker than the condition for a unique determination of the positions of nonzero elements in $\mathbf{X}$, at $k \in \mathbb{K}$, that will be considered next.

2.4. Coherence index

The coherence index of a matrix $\mathbf{A}$ is defined as the maximum absolute value of the normalized scalar product of its two different columns, that is, [25]

$$\mu = \max |\mu_{kl}|, \text{ for } k \neq l, \tag{6}$$

where the elements μ_{kl} are defined by

$$\mu_{kl} = \frac{1}{||\mathbf{a}_k||_2 ||\mathbf{a}_l||_2} \sum_{m=0}^{M-1} a_k(m) a_l^*(m) = \frac{\langle \mathbf{a}_k, \mathbf{a}_l \rangle}{||\mathbf{a}_k||_2 ||\mathbf{a}_l||_2} \tag{7}$$

and $a_k(m)$ is the element at the mth row and kth column of the measurement matrix $\mathbf{A}$ (whose kth column is denoted by $\mathbf{a}_k$). If the columns, $\mathbf{a}_k$, of the measurement matrix, $\mathbf{A}$, are energy normalized, $||\mathbf{a}_k||_2^2 = \sum_{m=0}^{M-1} |a_k(m)|^2 = 1$, then

$$\mu_{kl} = \sum_{m=0}^{M-1} a_k(m) a_l^*(m) = \langle \mathbf{a}_k, \mathbf{a}_l \rangle. \tag{8}$$

Notice that μ_{kl}, are then the elements of matrix $\mathbf{A}^H\mathbf{A}$.

The coherence index plays a crucial role in the measurement matrix design. The coherence index should be as small as possible, meaning that the incoherence is a desirable property for the measurement matrix [8]. With smaller values of the coherence index the matrix defined by $\mathbf{A}^H\mathbf{A}$ has lower off-diagonal elements and it is closer to the identity matrix.

3. Unique reconstruction

A K-sparse solution, $\mathbf{X}$, of the system (3), whose nonzero elements form the vector $\mathbf{X}_K$, is unique if all $\mathbf{A}_{2K}$ submatrices of the measurement matrix $\mathbf{A}$, corresponding to a $2K$-sparse signal, are such that all matrices $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ are invertible.

The contradiction will be used to prove this statement. This simple proof will be used as a basis for the derivation of the new limit for the sparsity. Assume that two different K-sparse solutions exist for the vector $\mathbf{X}$. Denote the nonzero elements of these solutions by $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$. The nonzero elements in $\mathbf{X}_K^{(1)}$ correspond to the positions $k \in \mathbb{K}_1$ in the original vector $\mathbf{X}$, while $\mathbf{X}_K^{(2)}$ contains the nonzero elements of vector $\mathbf{X}$, positioned at $k \in \mathbb{K}_2$. Assume that the solution is not unique and that both of these two vectors satisfy the measurement Eqs. (3) and (4), that is,

$$\mathbf{A}_K^{(1)} \mathbf{X}_K^{(1)} = \mathbf{y} \quad \text{and} \quad \mathbf{A}_K^{(2)} \mathbf{X}_K^{(2)} = \mathbf{y},$$

where $\mathbf{A}_K^{(1)}$ and $\mathbf{A}_K^{(2)}$ are submatrices of the measurement matrix $\mathbf{A}$ of size $M \times K$. They correspond to the nonzero elements in vectors $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$, respectively. We can rewrite these two equations by adding zeros at the corresponding zero positions of the other vector, as

$$\begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{X}_K^{(1)} \\ \mathbf{0}_K \end{bmatrix} = \mathbf{y} \quad \text{and} \quad \begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{0}_K \\ \mathbf{X}_K^{(2)} \end{bmatrix} = \mathbf{y}. \tag{9}$$

If we subtract these two equations we get

$$\begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{X}_K^{(1)} \\ -\mathbf{X}_K^{(2)} \end{bmatrix} = \mathbf{0}. \tag{10}$$

We arrived at the homogeneous system of equations. It is known that this system does not have a nonzero solution for the elements of $\mathbf{X}_K^{(1)}$ and $\mathbf{X}_K^{(2)}$ if the rank of matrix $\mathbf{A}_{2K} = \begin{bmatrix} \mathbf{A}_K^{(1)} & \mathbf{A}_K^{(2)} \end{bmatrix}$ is equal to $2K$, meaning that $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ is invertible. If all possible submatrices $\mathbf{A}_{2K}$ of the measurement matrix $\mathbf{A}$, for all possible combinations of nonzero element positions, are such that $\mathbf{A}_{2K}^H \mathbf{A}_{2K}$ are invertible then two distinct solutions, whose sparsity is K, cannot exist. This means that the solution of the compressive sensing problem is unique. Note that there are $\binom{N}{2K}$ submatrices $\mathbf{A}_{2K}$, and the combinatorial approach to this problem is not computationally feasible.

4. Review of the Gershgorin disk theorem

The matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ is invertible if its determinant is nonzero [29]. This condition is equivalent to the condition that all eigenvalues of matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$, for all possible combinations of $2K$ positions of nonzero elements, are nonzero. The eigenvalue/eigenvector relation for a matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ is defined by

$$(\mathbf{A}_{2K}^H\mathbf{A}_{2K})\mathbf{u} = \lambda\mathbf{u}, \tag{11}$$

where $\mathbf{u}$ denotes the eigenvector corresponding to the eigenvalue λ. Since the eigenvector belongs to the kernel of $\mathbf{A}_{2K}^H\mathbf{A}_{2K} - \lambda\mathbf{I}$ we can always assume that its maximum coordinate is equal to 1 (instead of the common choice to produce a unit energy eigenvector), that is $u_i = \max_j(|u_j|) = 1$ and $|u_j| \le 1$ for $j \ne i$.

For the columns $k \in \{k_1, k_2, \dots, k_{2K}\}$ of matrix $\mathbf{A}_{2K}$, selected from the columns of normalized matrix $\mathbf{A}$, the elements of matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ are denoted by

$$\mu_{k_ik_j} = \sum_{m=0}^{M-1} a_{k_i}(m)a^*_{k_j}(m) = \langle \mathbf{a}_{k_i}, \mathbf{a}_{k_j}\rangle, \tag{12}$$

for $i, j = 1, 2, \dots, 2K$. Now, we can rewrite the eigenvalue relation (11), in the element-wise form for the selected coordinate $u_i = 1$, as

$$\sum_j \mu_{k_ik_j}u_j = \lambda u_i = \lambda \quad \text{or} \quad \sum_{j,j\ne i} \mu_{k_ik_j}u_j = \lambda - \mu_{k_ik_i}.$$

From this relation we can conclude (Gershgorin Disc Theorem result)

$$|\lambda - \mu_{k_ik_i}| = \Big|\sum_{j,j\ne i}\mu_{k_ik_j}u_j\Big| \le \sum_{j,j\ne i}|\mu_{k_ik_j}u_j| \le \sum_{j,j\ne i}|\mu_{k_ik_j}|, \tag{13}$$

where the property $|u_j| \le 1$ for $j \ne i$ is used. Considering the eigenvalue λ as a variable and $\mu_{k_ik_j}$ as constants, we conclude that the last inequality, $|\lambda - \mu_{k_ik_i}| \le \sum_{i,j\ne i}|\mu_{k_ik_j}|$, describes a disc area in the complex domain of λ, with the center at $\mu_{k_ik_i}$ and a radius $\sum_{j,j\ne i}|\mu_{k_ik_j}|$. The disc described by the relation in (13) does not include the point $\lambda = 0$ if the radius is smaller than the distance of the center from the origin, that is, if

$$\mu_{k_ik_i} > \sum_{j,j\ne i}|\mu_{k_ik_j}|. \tag{14}$$

Therefore, if the condition in (14) is met, the matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ cannot assume a zero-valued eigenvalue, λ, and therefore it is invertible. Notice that $\mu_{k_ik_i} = 1$, for a normalized measurement matrix $\mathbf{A}$.

We have already concluded that the solution for a K-sparse vector is unique if the matrices $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ are invertible for all possible submatrices $\mathbf{A}_{2K}$. Note that the off-diagonal elements of $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ represent a subset of the off-diagonal elements of the matrix $\mathbf{A}^H\mathbf{A}$, that is

$$\{\mu_{k_ik_j} |\ k_i, k_j \in \{k_1, k_2, \dots, k_{2K}\}, j \ne i\} \\ \subset \{\mu_{kl} |\ k, l \in \{0, 1, \dots, N-1\}, k \ne l\}.$$

It means that the coherence μ of the measurement matrix $\mathbf{A}$ will be always greater than or equal to the coherence of any submatrix $\mathbf{A}_{2K}$, that is, $\max_{i,j,j\ne i}|\mu_{k_ik_j}| \le \max_{k,l,k\ne l}|\mu_{kl}| = \mu$, for $k_i, k_j \in \{k_1, k_2, \dots, k_{2K}\}$, $j \ne i$ and $k, l \in \{0, 1, \dots, N-1\}$, $k \ne l$. These two sets of indices are related as in (1).

The invertibility condition for all matrices $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$, and the unique solution for a K sparse vector $\mathbf{X}$, is achieved if $1 > (2K-1)\mu$ or

$$K < \frac{1}{2}(1 + \frac{1}{\mu}). \tag{15}$$

The proof of this classical coherence index-based uniqueness condition follows from (14) for the normalized matrix $\mathbf{A}^H\mathbf{A}$. The inequality

$$1 = \mu_{k_ik_i} > \sum_{j=1,j\ne i}^{2K}|\mu_{k_ik_j}| \tag{16}$$

is satisfied if $1 > (2K-1)\max_{i,j,j\ne i}|\mu_{k_ik_j}|$. Since all matrices $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ are submatrices of the matrix $\mathbf{A}^H\mathbf{A}$ then $\max_{i,j,j\ne i}|\mu_{k_ik_j}| \le \max_{l,k,l\ne k}|\mu_{kl}| = \mu$ holds. This means that $\sum_{j=1,j\ne i}^{2K}|\mu_{k_ik_j}| \le (2K-1)\mu$ and (16) is always satisfied if $1 > (2K-1)\mu$, producing (15).

5. Improved bound

The coherence index bound is pessimistic by definition, since it takes the worst possible value of μ_{kl}, over the whole matrix $\mathbf{A}^H\mathbf{A}$, which is equal to μ, and assigns it to each of $(2K-1)$ terms $\mu_{k_ik_j}$ in the sum in (16). This means that we may improve the coherence index-based bound in the Gershgorin disc theorem derivation using the sum of the $(2K-1)$ largest absolute values instead of using $(2K-1)$ times the largest absolute value μ, like in Stanković et al. [26], when the coherence index was analyzed.

Proposition: *A unique solution of the reconstruction problem, for a K sparse vector $\mathbf{X}$, exists if*

$$K < \frac{1}{2}\Big(1 + \frac{1}{\beta_{\mathbf{A}}(2K-1)}\Big), \tag{17}$$

where $\beta_{\mathbf{A}}(2K-1)$ is the mean of the $(2K-1)$ largest absolute values of the off-diagonal elements of the matrix$\mathbf{A}^H\mathbf{A}$ within one row/column.

The condition that a matrix $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ is invertible is equivalent to the condition that $\lambda = 0$ is not an eigenvalue of $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$. According to (14), this is the case when

$$1 > \max_i\{\sum_{j=1,j\ne i}^{2K}|\mu_{k_ik_j}|\} \tag{18}$$

holds for all possible combinations of $2K$ out of N indices, $\mathbb{K}_1 \cup \mathbb{K}_2 = \{k_1, k_2, \dots, k_{2K}\} \subset \{0, 1, \dots, N-1\}$.

In order to avoid combinatorial (NP hard) approach, we can use the largest values of this sum over the complete matrix $\mathbf{A}^H\mathbf{A}$. Denote the sorted absolute values of the elements, μ_{kl}, in the columns (or rows) of the matrix $\mathbf{A}^H\mathbf{A}$ by

$$s(k, p) = \text{sort}_l\{|\,\mu_{kl}\,|\},$$

such that $s(k, 1) \ge s(k, 2) \ge \dots \ge s(k, N)$. Then, having in mind that $\mathbf{A}_{2K}^H\mathbf{A}_{2K}$ are submatrices of $\mathbf{A}^H\mathbf{A}$, the condition in (18) will be satisfied if

$$1 > \max_k\{(2K-1)\frac{1}{2K-1}\sum_{p=1}^{2K-1}s(k, p)\} \tag{19}$$

holds. Using the notation

$$\beta_{\mathbf{A}}(2K-1) = \max_k\{\text{mean}\{s(k, p)|\ p = 1, 2\dots, 2K-1\}\},$$

the inequality in (19) can be written in the following form

$$1 > (2K-1)\beta_{\mathbf{A}}(2K-1), \tag{20}$$

producing (17), where $\beta_{\mathbf{A}}(2K-1)$ is the mean of the $(2K-1)$ largest absolute values of the off-diagonal elements of the matix $\mathbf{A}^H\mathbf{A}$ within one row/column.

The implicit inequality (17) is easily solved by checking for the sparsity values, $K = 1$, $K = 2$, and so on, until the inequality sign holds.

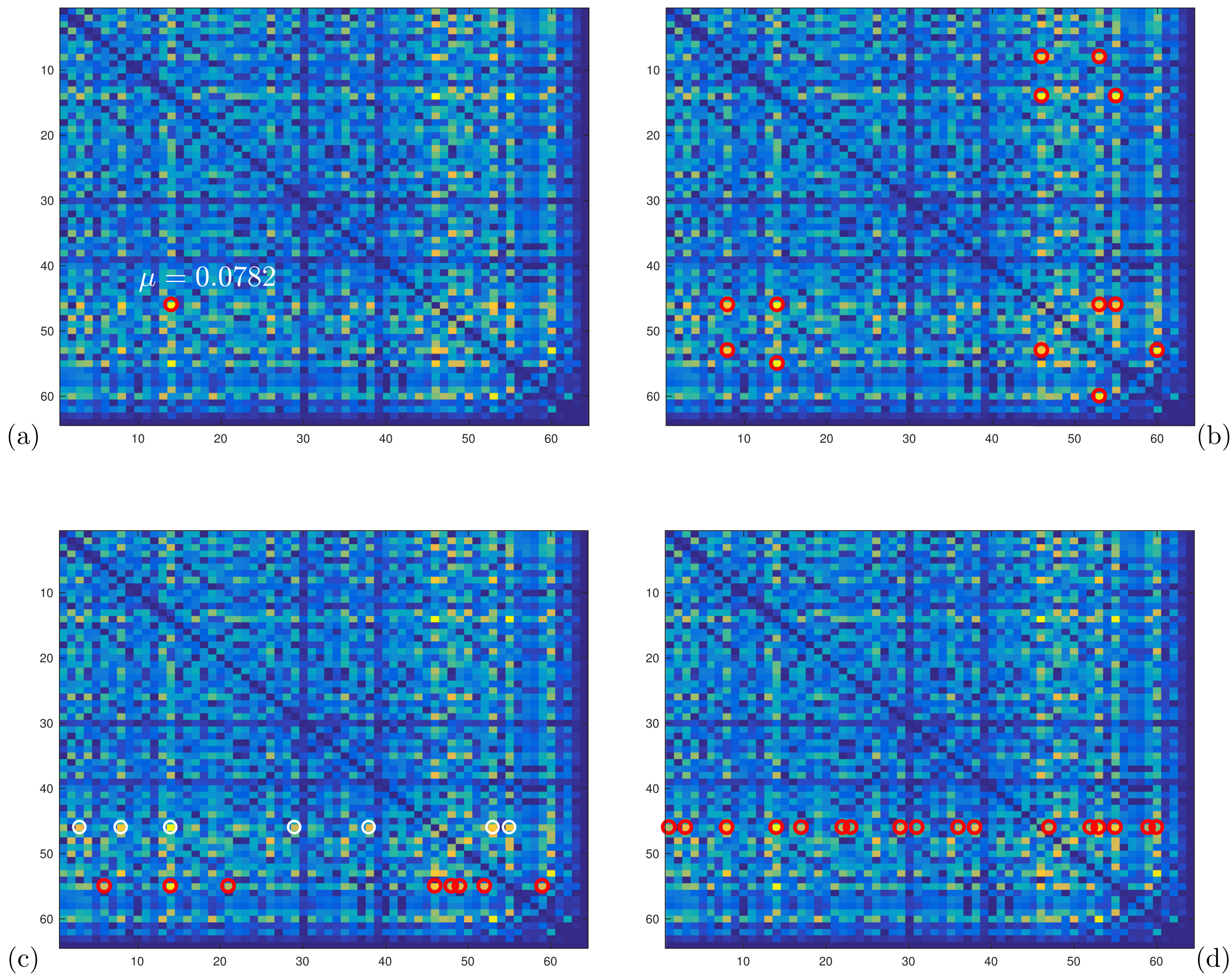

Fig. 1. The off-diagonal elements of matrix $\mathbf{A}^H\mathbf{A}$ used for the calculation of various bounds. (a) The coherence index, as the largest absolute off-diagonal element of matrix $\mathbf{A}^H\mathbf{A}$ (or the largest absolute element of matrix $\mathbf{A}^H\mathbf{A}-\mathbf{I}$), used in the sparsity bound in (15), marked with a red circle. (b) The largest absolute values of elements in $\mathbf{A}^H\mathbf{A}-\mathbf{I}$ used to calculate $\alpha_\mathbf{A}$ and the bound in (21). (c) The largest absolute values in $\mathbf{A}^H\mathbf{A}-\mathbf{I}$ used to calculate $\beta_\mathbf{A}(K-1)$ (encircled using a white line) and $\gamma_\mathbf{A}(K)$ (encircled using a red line) used in the bound in (22). (d) The largest absolute values in $\mathbf{A}^H\mathbf{A}-\mathbf{I}$ used to calculate $\beta_\mathbf{A}(2K-1)$ and the bound in (17). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5.1. Comparison of bounds

Next, we will compare the bound in (17) with other bounds derived in [26]. It is obvious that this new bound can improve the standard coherence index-based bound (15) since the maximum absolute value is always greater or equal to the mean of $(2K-1)$ largest absolute values, $\mu \geq \beta_\mathbf{A}(2K-1)$, that is

$$K < \frac{1}{2}(1+\frac{1}{\mu}) \leq \frac{1}{2}(1+\frac{1}{\beta_\mathbf{A}(2K-1)}).$$

Illustration of the values used in the calculation of the bound in (17) and the standard coherence index-based bound (15) is shown in Fig. 1(a) and (d).

The bound in (17) is obtained using the mean of the $(2K-1)$ largest absolute values of the off-diagonal elements of the matrix $\mathbf{A}^H\mathbf{A}$ *within one row/column* and it will therefore be always larger or equal to the bound obtained in [26] using the average of the $(2K-1)$ largest absolute values *within the whole matrix* $\mathbf{A}^H\mathbf{A}$ (Fig. 1(b)), denoted by $\alpha_\mathbf{A}$, that is

$$K < \frac{1}{2}(1+\frac{1}{\alpha_\mathbf{A}}) \leq \frac{1}{2}(1+\frac{1}{\beta_\mathbf{A}(2K-1)}). \tag{21}$$

The bound derived in this paper is compared with one derived in [26], when the maximum absolute values *within two different rows* are used (whose means are denoted by $\beta_\mathbf{A}(K-1)$ and $\gamma_\mathbf{A}(K)$, Fig. 1(c)), which is defined by

$$K < \frac{1+\beta_\mathbf{A}(K-1)}{\beta_\mathbf{A}(K-1)+\gamma_\mathbf{A}(K)}. \tag{22}$$

We cannot decisively conclude which one of the bounds in (17) or (22) is better since two different rows are used in the calculation of (22). In the examples that will be presented next, the inequality (17) produced higher sparsity bound than (22) in all considered cases.

All the previous bounds produce the same result for the equiangular tight frame (ETF) measurement matrices, when all $|\mu_{k_ik_j}| = \mu$ are equal for any $k_i \neq k_j$, and $\beta_\mathbf{A}(K-1) = \gamma_\mathbf{A}(K) = \alpha_\mathbf{A} = \mu$.

Finally, note that while the limit derived in [26] *guarantees successful reconstruction* using the matching pursuit approach, the relaxed condition derived in this paper *guarantees only the existence* of a unique solution.

6. Numerical examples

The derived limit for the sparsity was tested on several measurement matrices, including the partial graph Fourier transform (GFT) matrix, the partial DFT matrix, the partial DCT matrix, and a random Gaussian measurement matrix.

- For a partial DFT matrix $\mathbf{A}$ of dimension 124×128 the sparsity limit obtained with the standard coherence index relation (15) is $K < 16.63$. For the limits (21) and (22) we get $K < 16.63$ and $K < 19.20$, respectively. For the limit in (17) we get $K < 23.54$. The proposed result improves the classical coherence index bound for almost 50%.
- For a Gaussian measurement matrix $\mathbf{A}$ of dimension 900×1000 we get $K < 3.40$ as the classical limit and $K < 3.59$ and $K < 4.48$, as the bounds in (21) and (22) respectively. With (17) we get $K < 4.84$.
- For a partial DCT matrix of the size 124×128 we get $K < 9.05$, $K < 9.77$, $K < 12.47$, and $K < 15.11$, with the bounds defined by (15), (21), (22), and (17), respectively.
- For a partial GFT matrix of a graph with $N = 64$ vertices and 62 available graph signal samples, given in Stanković et al. [26], the classical coherence index relation produces $K < 6.89$. The bounds in (21) and (22) produce $K < 7.46$ and $K < 8.31$, while the bound in (17) produces $K < 9.22$, as illustrated in Fig. 1.

7. Conclusion

An improved bound for the reconstruction limit has been recently proposed based on the coherence index analysis. In this paper, this bound is further relaxed by considering the existence of the unique solution only and using the Gershgorin disc theorem.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] D.L. Donoho, Compressed sensing, IEEE Trans. Inf. Theory 52 (4) (2006) 1289–1306.
[2] S. Qaisar, R.M. Bilal, W. Iqbal, M. Naureen, S. Lee, Compressive sensing: from theory to applications, a survey, J. Commun. Netw. 15 (5) (Oct. 2013) 443–456, doi:10.1109/JCN.2013.000083.
[3] M. Rani, S.B. Dhok, R.B. Deshmukh, A systematic review of compressive sensing: concepts, implementations and applications, IEEE Access 6 (2018) 4875–4894, doi:10.1109/ACCESS.2018.2793851.
[4] L. Stanković, E. Sejdić, S. Stanković, M. Daković, I. Orović, A tutorial on sparse signal reconstruction and its applications in signal processing, Circuits, Syst., Signal Process. 38 (3) (2019) 1206–1263.
[5] E.J. Candès, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inf. Theory 52 (2) (2006) 489–509.
[6] J.A. Tropp, A.C. Gilbert, Signal recovery from random measurements via orthogonal matching pursuit, IEEE Trans. Inf. Theory 53 (12) (2007) 4655–4666.
[7] S. Stanković, I. Orović, L. Stanković, An automated signal reconstruction method based on analysis of compressive sensed signals in noisy environment, Signal Process. 104 (Nov 2014) 43–50.
[8] L. Stanković, Digital Signal Processing with Applications: Adaptive Systems, Time-Frequency Analysis, Sparse Signal Processing, CreateSpace Independent Publishing Platform, 2015.
[9] I. Orović, V. Papić, C. Ioana, X. Li, S. Stanković, Compressive sensing in signal processing: algorithms and transform domain formulations, Math. Probl. Eng. 2016 (2016) 7616393, doi:10.1155/2016/7616393.
[10] L. Stanković, M. Daković, S. Vujović, Adaptive variable step algorithm for missing samples recovery in sparse signals, IET Signal Process. 8 (3) (2014) 246–256.
[11] L. Stanković, M. Daković, S. Vujović, Reconstruction of sparse signals in impulsive disturbance environments, Circuits, Syst. Signal Process. 36 (2) (2017) 767–794.
[12] L. Stankovic, M. Brajovic, Analysis of the reconstruction of sparse signals in the DCT domain applied to audio signals, IEEE/ACM Trans. Audio Speech Lang. Process. 26 (7) (July 2018) 1216–1231, doi:10.1109/TASLP.2018.2819819.
[13] I. Stanković, I. Orović, M. Daković, S. Stanković, Denoising of sparse images in impulsive disturbance environment, Multimed. Tools Appl. 77 (5) (2018) 5885–5905.
[14] L. Stanković, D. Mandic, M. Daković, M. Brajović, B. Scalzo-Dees, S. Li, A.G. Constantinides, Data Analytics on Graphs, NOW Publishers, February 2021.
[15] E.J. Candès, The restricted isometry property and its implications for compressed sensing, C.R. Math. 346 (9–10) (May 2008) 589–592.
[16] T. Zhang, Sparse recovery with orthogonal matching pursuit under RIP, IEEE Trans. Inf. Theory 57 (9) (2011) 6215–6221.
[17] L. Stanković, M. Daković, On the uniqueness of the sparse signals reconstruction based on the missing samples variation analysis, Math. Probl. Eng. (2015) 14, doi:10.1155/2015/629759. Article ID 629759
[18] L. Stankovic, On the ISAR image analysis and recovery with unavailable or heavily corrupted data, IEEE Trans. Aerosp. Electron. Syst. 51 (3) (July 2015) 2093–2106.
[19] J. Ender, On compressive sensing applied to radar, Signal Process. 90 (5) (May 2010) 1402–1414.
[20] P. Flandrin, P. Borgnat, Time-frequency energy distributions meet compressed sensing, IEEE Trans. Signal Process. 58 (6) (June 2010) 2974–2982.
[21] L. Stankovic, I. Orovic, S. Stankovic, M. Amin, Compressive sensing based separation of non-stationary and stationary signals overlapping in time-frequency, IEEE Trans. Signal Process. 61 (18) (Sept. 2013) 4562–4572.
[22] I. Orović, S. Stanković, T. Chau, C.M. Steele, E. Sejdić, Time-frequency analysis and hermite projection method applied to swallowing accelerometry signals, EURASIP J. Adv. Signal Process. 2010 (2010) 7. Article ID 323125
[23] G. Hua, Y. Hiang, G. Bi, When compressive sensing meets data hiding, IEEE Signal Process. Lett. 23 (4) (2016) 473–477.
[24] J.W. Choi, B. Shim, Y. Ding, B. Rao, D.I. Kim, Compressed sensing for wireless communications: useful tips and tricks, IEEE Commun. Surv. Tutor. 19 (3) (2017) 1527–1550, doi:10.1109/COMST.2017.2664421.
[25] L. Stanković, D. Mandic, M. Daković, I. Kisil, Demystifying the coherence index in compressive sensing, IEEE Signal Process. Mag. 37 (1) (2020) 152–162.
[26] L. Stanković, M. Brajović, D. Mandic, I. Stanković, M. Daković, Improved coherence index-based bound in compressive sensing, IEEE Signal Process. Lett. 28 (2021) 1110–1114, doi:10.1109/LSP.2021.3084559.
[27] L. Stanković, The support uncertainty principle and the graph Rihaczek distribution: revisited and improved, IEEE Signal Process. Lett. 27 (2020) 1030–1034, doi:10.1109/LSP.2020.3000686.
[28] M. Elad, A.M. Bruckstein, A generalized uncertainty principle and sparse representation in pairs of bases, IEEE Trans. Inf. Theory 48 (9) (Sept. 2002) 2558–2567, doi:10.1109/TIT.2002.801410.
[29] R.S. Varga, Geršgorin and his Circles, Springer-Verlag, 2004.