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The Schur decomposition of discrete Sine and Cosine transformations of type IV

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ABSTRACT

Eigendecomposition of discrete sine and cosine transformations of type IV is the main objective of this study. A closed form formula for eigenvectors is derived using the property that a square of the transform matrix is proportional to the identity matrix. Odd and even transform sizes are considered separately. In both cases, it has been demonstrated that the eigenvectors can be obtained by a straightforward modification of the columns of the transformation matrix. Comparing this approach with the related works, it provides a simple explanation of the eigendecomposition with a reliable numerical stability. Additionally, these eigenvectors can be used to obtain the eigendecomposition of the counterpart offset discrete Fourier transform.

1. Introduction

The main contribution of this paper is in a novel approach to the eigendecomposition of discrete sine and cosine transforms of type IV. Broadly speaking, all types of discrete trigonometric transforms (DTTs) are deeply rooted in the discrete offset Fourier transform, also known as the generalized Fourier transform (GDFT) [1–9]. Thus, the offset Fourier transforms are expected to be in charge of providing the eigendecomposition of the DTTs. Following this point of view, some algorithms have been introduced to form particular eigendecompositions of the discrete sine and cosine transforms of type IV from the counterpart's offset Fourier transform [10–12]. In this paper, using the proposed approach, we show that the DCT-IV and DST-IV are so well structured that one may directly provide a basis in an analytic form. Furthermore using this result, an eigendecomposition of the counterpart's offset discrete Fourier transform is determined.

Denote by $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ the transformation matrix of order n for the discrete sine and cosine transforms of type IV, with elements

$$S(k,l) = \sin\frac{(2k+1)(2l+1)\pi}{4n}, \tag{1}$$

$$C(k,l) = \cos\frac{(2k+1)(2l+1)\pi}{4n}, \tag{2}$$

where $k, l = 0, 1, \ldots, n-1$ are row and column index, respectively.

Eigenvalues of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ are analytically derived in [13] and they are

$$\lambda_1 = \sqrt{\tfrac{n}{2}}, \qquad \text{with multiplicity: } \tfrac{n}{2} \text{ for even } n \text{ and } \tfrac{n+1}{2} \text{ for odd } n, \tag{3}$$

$$\lambda_2 = -\sqrt{\tfrac{n}{2}}, \qquad \text{with multiplicity: } \tfrac{n}{2} \text{ for even } n \text{ and } \tfrac{n-1}{2} \text{ for odd } n. \tag{4}$$

Throughout this paper, we use $\mathbf{T}$ for both transform matrices $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$. The strategy relies deeply on the following key identity,

$$\mathbf{T}^2 = \frac{n}{2}\mathbf{I}_n, \tag{5}$$

where $\mathbf{I}_n$ is the identity matrix. By rewriting identity (5), we get

$$\mathbf{T}(\mathbf{T} + \sqrt{\tfrac{n}{2}}\mathbf{I}_n) = \sqrt{\tfrac{n}{2}}(\mathbf{T} + \sqrt{\tfrac{n}{2}}\mathbf{I}_n)$$

$$\mathbf{T}(\mathbf{T} - \sqrt{\tfrac{n}{2}}\mathbf{I}_n) = -\sqrt{\tfrac{n}{2}}(\mathbf{T} - \sqrt{\tfrac{n}{2}}\mathbf{I}_n).$$

Based on these two identities, any vector in the image of the transforms $\mathbf{T} + \sqrt{\tfrac{n}{2}}\mathbf{I}_n$ and $\mathbf{T} - \sqrt{\tfrac{n}{2}}\mathbf{I}_n$ is an eigenvector of $\mathbf{T}$, that is

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$$\mathbb{R}^n \overset{\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n}{\Longrightarrow} \text{Eigenvectors of } \mathbf{T} \text{ corresponded to } \sqrt{\tfrac{n}{2}}$$

$$\mathbb{R}^n \overset{\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n}{\Longrightarrow} \text{Eigenvectors of } \mathbf{T} \text{ corresponded to } -\sqrt{\tfrac{n}{2}}.$$

The first relation confirms that any column of $\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n$ is an eigenvector of $\mathbf{T}$, corresponding to the eigenvalue $\sqrt{\frac{n}{2}}$, while the second relation states that any column of $\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n$ is an eigenvector of $\mathbf{T}$, corresponding to the eigenvalue $-\sqrt{\frac{n}{2}}$. We can also write

$$\operatorname{rank}(\mathbf{T}+\sqrt{\tfrac{n}{2}}\mathbf{I}_n)+\operatorname{rank}(\mathbf{T}-\sqrt{\tfrac{n}{2}}\mathbf{I}_n)=n.$$

Hence, the $2n$-columns of matrices $\mathbf{T}\pm\sqrt{\frac{n}{2}}\mathbf{I}_n$ may certainly serve as a basis consisting of the eigenvectors of $\mathbf{T}$. To make a basis, all that is needed to be done is an appropriate reduction of the columns. Since there are only two different eigenvalues, there is an infinite number of bases available. Thus, it would be appropriate to find an exclusive rule that governs the decision-making on which columns are eligible to form the eigenbasis. Using linear algebra techniques, in this paper, we will propose a rule for any size of the transformation matrix.

This paper is organized as follows. In Section 2, we briefly review some related results in the literature on the eigenspaces of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ according to excerpt of papers [10–12]. They are based on the sampling theory. Then we proceed to the results proposed in this paper. Depending upon whether the order n is even or odd, two completely different approaches should be followed. The odd and even cases are separately discussed in Sections 3 and 4. The required details to figure out $\mathbf{C}_{(4)}$ are completely similar to $\mathbf{S}_{(4)}$. Due to this, we only refer to the benchmarks. In Section 5, a two-sided way is defined between the pair $(\mathbf{C}_{(4)},\mathbf{S}_{(4)})$ and its offset DFT counterpart. Based on these results, it is explained how all eigenstructures points are converted. The numerical stability of the proposed bases is analyzed in Section 6. In the Appendix, all derivations required in the main sections of the paper are provided. They are based on a number of trigonometric identities [14].

This study provides an eigendecomposition of type IV DCT and DST, subsequently guiding us towards the eigendecomposition of the counterpart's offset DFT. The offset DFT, distinguished by its heightened flexibility in comparison to the original DFT, holds relevance in diverse areas such as filter design, signal representation, and expeditious DFT computation. For a comprehensive understanding, the interpretation of the corresponding eigendecomposition proves to be of immense value.

2. Related works

The offset DFT of order n, with parameters a and b is a $n\times n$ matrix whose entries are defined as

$$\mathbf{G}_{a,b}(k,l)=\frac{1}{\sqrt{n}}\exp\left(\tfrac{2\pi i}{n}(k-a)(l-b)\right).$$

A set of the offset DFT eigenvectors can be (approximately) obtained by sampling the eigenfunctions of the continuous offset FT. A Hermite-like eigenvector set of the offset DFT$_n$, when is $a+b$ is an integer, is given by [10, p. 2035]

$$v_{q;a,b}(m)=e^{i\pi\frac{(b-a)m}{n}}\sum_{p=-\infty}^{+\infty}(-1)^{(a+b)p}e^{-\frac{\pi\left(m+pn-\frac{a+b}{2}\right)^2}{n}}$$
$$\times H_q\Big((m+pn-\frac{a+b}{2})\sqrt{\frac{2\pi}{n}}\Big),$$

where $v_{q;a,b}(m)$ are the elements of the eigenvector $\mathbf{v}_{q;a,b}$, $H_q(t)$ is the Hermit polynomial of order q, $m=0,1,\dots,n-1$, $q=0,1,\dots,n-2,n_1$, where

$$n_1=\begin{cases} n & \text{when } n+a+b \text{ is even}\\ n-1 & \text{when } n+a+b \text{ is odd.}\end{cases}$$

Let $a=b=-\frac{1}{2}$. It has been shown in [11] that the offset transform $\mathbf{G}_{a,b}$ satisfies $\mathbf{G}^2_{a,b}=-\mathbf{J}_n$, where $\mathbf{J}_n$ is the antidigonal matrix. Furthermore, we have $\mathbf{G}^4_{a,b}=\mathbf{I}_n$. Due to this fact, the eigenvalues of $\mathbf{G}_{a,b}$ are equal to $\{\pm1,\pm i\}$. Any eigenvector $\mathbf{v}$, corresponding to the eigenvalues $\pm i$, satisfies $\mathbf{J}_n\mathbf{v}=\mathbf{v}$. If $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue ±1, then $\mathbf{J}_n\mathbf{v}=-\mathbf{v}$.

The eigenvectors of both transforms $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ are extracted from the offset $\mathbf{G}_{a,b}$ of order $2n$ [11, p. 2040]. The eigenvectors $\mathbf{v}_{q;\mathbf{C}_{(4)}}$ of $\mathbf{C}_{(4)}$ and $\mathbf{v}_{q;\mathbf{S}_{(4)}}$ of $\mathbf{S}_{(4)}$ are as follows,

$$v_{q;\mathbf{C}_{(4)}}(m)=\sum_{p=-\infty}^{+\infty}(-1)^p e^{-\frac{\pi\left(m+2pn+\frac{1}{2}\right)^2}{2n}}\times H_{2q}\Big((m+2pn+\frac{1}{2})\sqrt{\frac{\pi}{n}}\Big)$$

$$v_{q;\mathbf{S}_{(4)}}(m)=\sum_{p=-\infty}^{+\infty}(-1)^p e^{-\frac{\pi\left(m+2pn+\frac{1}{2}\right)^2}{2n}}\times H_{2q+1}\Big((m+2pn+\frac{1}{2})\sqrt{\frac{\pi}{n}}\Big),$$

where $m=0,1,\dots,n-1$ and $q=0,\cdots,n-1$.

The Hermit-like set of eigenvectors of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ is not orthogonal. By theory of commuting matrices [15], an alternative method is presented in [10] in order to fix the orthogonality property. Let us consider the following tridiagonal-like matrix

$$\mathbf{S}=\begin{bmatrix} 2\cos\omega & 1 & 0 & \cdots & -1\\ 1 & 2\cos3\omega & 1 & \cdots & 0\\ 0 & 1 & \ddots & & \vdots\\ \vdots & \vdots & & & 1\\ -1 & 0 & \cdots & 1 & 2\cos(2n-1)\omega \end{bmatrix}.$$

It has been shown that

1. Matrix $\mathbf{S}$ has n-distinct eigenvalues and so the set of the corresponded eigenvectors is orthogonal. To determine these eigenvectors, the matrix equation $(\mathbf{S}-\lambda\mathbf{I}_n)\mathbf{v}=0$ needs to be solved. This problem is equivalent to the following linear difference equation
$$v(k+1)+\Big(2\cos(2k+1)\frac{\pi}{n}-\lambda\Big)v(k)-v(k-1)=0,$$
when $k=0,1,\dots,n-1$, assuming that $v(-1)=v(n-1)$ and $v(n)=v(0)$. By a sampled Hermite-Gaussian sequence, a good approximation of the eigenvector $\mathbf{v}$ is obtained.
2. Matrix $\mathbf{S}$ commutes with the offset DFT counterparts of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ (i.e., when the parameters $a=b=-\frac{1}{2}$). This fact leads to the conclusion that any eigenvector of $\mathbf{S}$ would be an eigenvector of $\mathbf{G}_{a,b}$ as well. This approach provides an orthogonal basis consisting of the eigenvectors for $\mathbf{G}_{a,b}$. Then, one may extract an orthogonal basis consisting of the eigenvectors for both $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$.

A brief examination of both approaches reveals the existence of two significant gaps. Firstly, access to the eigenvectors of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ remains largely approximate, due to their representation as infinite series. Additionally, the indirect access facilitated through the offset DFT increased complexity. This paper, in contrast, presents a distinct approach by outlining a direct method to derive a closed form and precise formula for the eigenvectors.

3. Eigenvectors of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ of an odd order

In equations (3) and (4), the eigenvalues and their corresponding multiplicities for $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ are given. In this section, we prove the following main theorem which proposes a unified approach to determine the eigendecomposition when the order n of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ is odd.

Theorem 1. *Let us consider the transform* $\mathbf{T}$.

1. *Columns of matrix* $\left(\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ *are the eigenvectors of* $\mathbf{T}$ *corresponding to the eigenvalue* $\sqrt{\frac{n}{2}}$. *Moreover, odd columns of* $\left(\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ *form a linearly independent set.*
2. *Columns of matrix* $\left(\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ *are the eigenvectors of* $\mathbf{T}$ *corresponding to the eigenvalue* $-\sqrt{\frac{n}{2}}$. *Moreover, even columns of* $\left(\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ *form a linearly independent set.*
3. *Let* $\mathbf{V}$ *be a square* $n\times n$ *matrix whose first* $\frac{n+1}{2}$ *columns are just the odd columns of* $\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n$ *and the rest of* $\frac{n-1}{2}$ *columns are the even columns of* $\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}$. *We have then,*

   $$\mathbf{T}=\mathbf{V}\mathbf{D}\mathbf{V}^{-1},$$

   where $\mathbf{D}$ *is a diagonal matrix given by direct sum*

   $$\mathbf{D}=\left(\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n+1}{2}}\right)\oplus\left(-\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n-1}{2}}\right).$$

To prove this theorem, some points need to be combined together. Let $\mathbf{T}_o$ (resp. $\mathbf{T}_e$) be the principal submatrix of $\mathbf{T}$ whose entries are just in the odd rows and odd columns (resp. even rows and even columns). Based on the derivations given in the Appendix, the square of $\mathbf{T}_o$ has the same formula in both cases of $\mathbf{T}=\mathbf{C}_{(4)}$ and $\mathbf{T}=\mathbf{S}_{(4)}$ and it is equal to

$$\mathbf{T}_o^2=\frac{1}{4}\left[\sec\frac{(k-l)\pi}{n}\right]_{k,l=0}^{\frac{n-1}{2}}+\frac{n}{4}\mathbf{I}_{\frac{n+1}{2}}. \quad (6)$$

Lemma 2. *Let us consider the principal submatrices* $\mathbf{T}_o$ *and* $\mathbf{T}_e$.

1. *Let* λ *be a scalar with* $|\lambda|\neq\sqrt{\frac{n}{2}}$. *Then* λ *is an eigenvalue of* $\mathbf{T}_o$ *if and only if* $-\lambda$ *is eigenvalue of* $\mathbf{T}_e$. *Moreover*

   $$\mathrm{Mult}_{\mathbf{T}_o}(\lambda)=\mathrm{Mult}_{\mathbf{T}_e}(-\lambda).$$

2. *The eigenvalues of* $\mathbf{T}_o$ *are contained in the interval* $(-\sqrt{\frac{n}{2}},\sqrt{\frac{n}{2}}]$. *Moreover,* $\sqrt{\frac{n}{2}}$ *is an eigenvalue of* $\mathbf{T}_o$ *whose multiplicity is 1.*
3. *The eigenvalues of* $\mathbf{T}_e$ *are contained in the interval* $(-\sqrt{\frac{n}{2}},\sqrt{\frac{n}{2}})$.

Proof. There is a permutation matrix $\mathbf{P}$ and a rectangular matrix $\mathbf{X}$ such that

$$\mathbf{P}^t\mathbf{T}\mathbf{P}=\begin{bmatrix}\mathbf{T}_o & \mathbf{X}\\ \mathbf{X}^t & \mathbf{T}_e\end{bmatrix}. \quad (7)$$

Using the identity $(\mathbf{P}^t\mathbf{T}\mathbf{P})^2=\frac{n}{2}\mathbf{I}_n$, we get the following identities:

$$\begin{aligned}\mathbf{X}\mathbf{X}^t&=\frac{n}{2}\mathbf{I}_{\frac{n+1}{2}}-\mathbf{T}_o^2,\\ \mathbf{T}_o\mathbf{X}^t&=-\mathbf{X}_e^t,\\ \mathbf{X}^t\mathbf{X}&=\frac{n}{2}\mathbf{I}_{\frac{n-1}{2}}-\mathbf{T}_e^2,\\ \mathbf{T}_e\mathbf{X}&=-\mathbf{X}\mathbf{T}_e.\end{aligned}$$

Suppose that λ is an eigenvalue of $\mathbf{T}_o$ with $|\lambda|\neq\frac{n}{2}$. The first identity confirms that $\mathbf{X}^t$ never vanishes on any eigenvector of $\mathbf{T}_o$ corresponding to λ. Using this point and applying the second identity, we observe that $-\lambda$ is an eigenvalue of $\mathbf{T}_e$. Furthermore, $\mathbf{X}^t$ is a one-to-one linear mapping from $\mathrm{Eigen}_{\mathbf{T}_o}(\lambda)$ onto $\mathrm{Eigen}_{\mathbf{T}_e}(-\lambda)$. Similarly, from the third and the fourth identity, we may say that $\mathbf{X}$ is a one-to-one linear mapping from $\mathrm{Eigen}_{\mathbf{T}_e}(-\lambda)$ onto $\mathrm{Eigen}_{\mathbf{T}_o}(\lambda)$.

By the interlacing property, the eigenvalues of both matrices $\mathbf{T}_o$ and $\mathbf{T}_e$ are contained in the interval $[-\sqrt{\frac{n}{2}},\sqrt{\frac{n}{2}}]$. We have that

$$\mathrm{Tr}(\mathbf{P}^t\mathbf{T}\mathbf{P})=\mathrm{Tr}(\mathbf{T})=\sqrt{\frac{n}{2}}=\mathrm{Tr}(\mathbf{T}_o)+\mathrm{Tr}(\mathbf{T}_e)=\sum_{\mathrm{Eigen}(\mathbf{T}_o)}\lambda+\sum_{\mathrm{Eigen}(\mathbf{T}_e)}\lambda.$$

We may conclude that $\sqrt{\frac{n}{2}}$ is an eigenvalue of $\mathbf{T}_o$. Since, the entries of $\mathbf{T}_o^2$ are all positive, the Perron-Frobenius theorem confirms that the largest eigenvalue of $\mathbf{T}_o^2$, which is just $\frac{n}{2}$, is repeated only once. Therefore we get both claims of the first item.

Suppose that either $-\sqrt{\frac{n}{2}}$ or $\sqrt{\frac{n}{2}}$ are in the set of eigenvalues of $\mathbf{T}_e$. By simultaneously using of the first part and $\mathrm{Tr}\left(\mathbf{P}^t\mathbf{T}\mathbf{P}\right)=\sqrt{\frac{n}{2}}$, we may conclude that $\frac{n}{2}$ repeats at least twice in the eigenvalues of $\mathbf{T}_o^2$ which is impossible. □

Proof of Theorem 1. By the identity $\mathbf{T}^2=\frac{n}{2}\mathbf{I}_n$, we may write

$$\mathbf{T}\left(\mathbf{T}+\sqrt{\tfrac{n}{2}}\mathbf{I}_n\right)=\sqrt{\tfrac{n}{2}}\left(\mathbf{T}+\sqrt{\tfrac{n}{2}}\mathbf{I}_n\right) \quad (8)$$

$$\mathbf{T}\left(\mathbf{T}-\sqrt{\tfrac{n}{2}}\mathbf{I}_n\right)=-\sqrt{\tfrac{n}{2}}\left(\mathbf{T}-\sqrt{\tfrac{n}{2}}\mathbf{I}_n\right). \quad (9)$$

Therefore, the column vectors of matrices $\left(\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ and $\left(\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n\right)$ are eigenvectors of $\mathbf{T}$ corresponding to the eigenvalues $\sqrt{\frac{n}{2}}$ and $-\sqrt{\frac{n}{2}}$ respectively. To prove the claim, we have to show that both rectangular matrices $\left(\mathbf{T}+\sqrt{\frac{n}{2}}\mathbf{I}_n\right)\mathbf{I}_o^c$ and $\left(\mathbf{T}-\sqrt{\frac{n}{2}}\mathbf{I}_n\right)\mathbf{I}_e^c$ enjoy a left inverse. Here, $\mathbf{I}_o^c$ and $\mathbf{I}_e^c$ stands for odd and even columns of the identity matrix, respectively. To do this, it is enough to check that the following square matrices $\mathbf{G}$ and $\mathbf{H}$,

$$\mathbf{G}=\mathbf{T}_e+\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n+1}{2}},\qquad \mathbf{H}=\mathbf{T}_o-\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n-1}{2}},$$

are invertible. This is equivalent to the verification that $-\sqrt{\frac{n}{2}}$ and $\sqrt{\frac{n}{2}}$ are not contained in $\mathrm{Eigen}(\mathbf{T}_e)$ and $\mathrm{Eigen}(\mathbf{T}_o)$ respectively. Lemma 2 confirms that they are valid. □

4. Eigenvectors of $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ of an even order

This section deals with the eigendecomposition of $\mathbf{S}_{(4)}$ and $\mathbf{C}_{(4)}$ when their order is even. The procedure is different from what has been done in the odd case. Let $\mathbf{Q}$ be an $n\times\frac{n}{2}$ matrix whose columns are given by

$$\mathbf{Q}=\left[\mathbf{e}_1+\mathbf{e}_2\quad \mathbf{e}_3+\mathbf{e}_4\quad\cdots\quad \mathbf{e}_{n-2}+\mathbf{e}_n\right], \quad (10)$$

where $\{\mathbf{e}_k\}_{k=1}^n$ is the standard basis of $\mathbb{R}^n$.

Theorem 3. *Let* $\mathbf{Q}$ *be the rectangular matrix introduced in (10) and*

$$\mathbf{T}^+=\left(\mathbf{T}+\sqrt{\tfrac{n}{2}}\mathbf{I}\right)\mathbf{Q},\qquad \mathbf{T}^-=\left(\mathbf{T}-\sqrt{\tfrac{n}{2}}\mathbf{I}\right)\mathbf{Q}.$$

1. *Columns of* $\mathbf{T}^+$ *are eigenvectors of* $\mathbf{T}$ *corresponding to* $\sqrt{\frac{n}{2}}$. *Moreover they form a linearly independent set.*
2. *Columns of* $\mathbf{T}^-$ *are eigenvectors of* $\mathbf{T}$ *corresponding to* $-\sqrt{\frac{n}{2}}$. *Moreover they form a linearly independent set.*
3. *Let* $\mathbf{V}$ *be an* $n\times n$ *matrix obtained as*

   $$\mathbf{V}=\left[\mathbf{T}^+\quad \mathbf{T}^-\right].$$

   Then we have

   $$\mathbf{T}=\mathbf{V}\mathbf{D}\mathbf{V}^{-1},$$

   where the diagonal matrix $\mathbf{D}$ *is defined by*

   $$\mathbf{D}=\left(\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n}{2}}\right)\oplus\left(-\sqrt{\tfrac{n}{2}}\mathbf{I}_{\frac{n}{2}}\right).$$

To prove this theorem, we apply the following lemma whose details are given in the Appendix.

Lemma 4. *Let* $\mathbf{G}$ *be a square of* $\mathbf{Q}^t\mathbf{T}\mathbf{Q}$*. Then*

$$\text{Eigen}(\mathbf{G}) = \left\{ n - n\cos\frac{\pi}{2n}, n + n\cos\frac{\pi}{2n} \right\}. \tag{11}$$

Proof of Theorem 3. Note that any column of $\left(\mathbf{T} + \sqrt{\frac{n}{2}}\mathbf{I}_n\right)\mathbf{Q}$ is just a linear combination of the columns of $\mathbf{T} + \sqrt{\frac{n}{2}}\mathbf{I}_n$. Thus, we should only check that both matrices $\mathbf{T}^+$ and $\mathbf{T}^-$ enjoy a left inverse. To this aim, it is enough to show that $\mathbf{Q}^t\mathbf{T}^+$ and $\mathbf{Q}^t\mathbf{T}^-$ are invertible. Based on Lemma 4

$$\sqrt{\frac{n}{2}} \notin \text{Eigen}(\mathbf{Q}^t\mathbf{T}^+), \qquad -\sqrt{\frac{n}{2}} \notin \text{Eigen}(\mathbf{Q}^t\mathbf{T}^-)$$

and so we get the desired result. □

5. Eigenvectors of the offset DFT with parameters $a = b = -\frac{1}{2}$

At the end, we will present a short discussion on the recovering all points of the eigenstructure of the offset DFT $\mathbf{G}_{a,b}$ (eigenvalues, multiplicities and eigenvectors) from the pair $(\mathbf{C}_{(4)}, \mathbf{S}_{(4)})$.

Let us put $a = b = -\frac{1}{2}$ and consider the offset DFT with parameters a, b for even n,

$$\mathbf{G}_{a,b} = \frac{1}{\sqrt{n}}\exp\left(\frac{2\pi i}{n}(k+\frac{1}{2})(l+\frac{1}{2})\right).$$

Let $\mathbf{E} = (g_{kl})$ be an $\frac{n}{2}\times\frac{n}{2}$ principal submatrix being equal to the first quarter of $\mathbf{G}_{a,b}$ and put $\overline{\mathbf{E}} = (\overline{g_{kl}})$. It can straightforwardly be checked that,

$$\sqrt{\frac{2}{n}}\mathbf{C}_{(4)} = \mathbf{E} + \overline{\mathbf{E}},$$

$$i\sqrt{\frac{2}{n}}\mathbf{S}_{(4)} = \mathbf{E} - \overline{\mathbf{E}},$$

$$\mathbf{G}_{a,b} = \begin{bmatrix} \mathbf{E} & -\overline{\mathbf{E}}\mathbf{J}_{\frac{n}{2}} \\ -\mathbf{J}_{\frac{n}{2}}\overline{\mathbf{E}} & \mathbf{J}_{\frac{n}{2}}\mathbf{E}\mathbf{J}_{\frac{n}{2}} \end{bmatrix},$$

where $\mathbf{J}_{\frac{n}{2}}$ is an antidigonal matrix of size $\frac{n}{2}$ whose nonzero entries are all 1. Using these identities, one may find the eigenvecotrs of $\mathbf{G}_{a,b}$ by a suitable arrangement of the values of the $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ eigenvectors. To do this, assume that $\mathbf{v}$ is in $\mathbb{R}^n$ and put,

$$\mathbf{v}' = [\mathbf{v} \quad -\mathbf{J}_{\frac{n}{2}}\mathbf{v}] = [v_0, \dots, v_{n-1}, -v_{n-1}, \dots, -v_0]^t,$$

$$\mathbf{v}'' = [\mathbf{v} \quad \mathbf{J}_{\frac{n}{2}}\mathbf{v}] = [v_0, \dots, v_{n-1}, v_{n-1}, \dots, v_0]^t.$$

We have then

$$\begin{bmatrix} \mathbf{E} & -\overline{\mathbf{E}}\mathbf{J}_{\frac{n}{2}} \\ -\mathbf{J}_{\frac{n}{2}}\overline{\mathbf{E}} & \mathbf{J}_{\frac{n}{2}}\mathbf{E}\mathbf{J}_{\frac{n}{2}} \end{bmatrix}\mathbf{v}' = \begin{bmatrix} \sqrt{\frac{2}{n}}\mathbf{C}_{(4)}\mathbf{v} \\ -\sqrt{\frac{2}{n}}\mathbf{J}_{\frac{n}{2}}\mathbf{C}_{(4)}\mathbf{v} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{E} & -\overline{\mathbf{E}}\mathbf{J}_{\frac{n}{2}} \\ -\mathbf{J}_{\frac{n}{2}}\overline{\mathbf{E}} & \mathbf{J}_{\frac{n}{2}}\mathbf{E}\mathbf{J}_{\frac{n}{2}} \end{bmatrix}\mathbf{v}'' = \begin{bmatrix} i\sqrt{\frac{2}{n}}\mathbf{S}_{(4)}\mathbf{v} \\ i\sqrt{\frac{2}{n}}\mathbf{J}_{\frac{n}{2}}\mathbf{S}_{(4)}\mathbf{v} \end{bmatrix}.$$

Therefore we get the following correspondences:

Eigenvectors of $\mathbf{C}_{(4)}$

→ Eigenvectors of $\mathbf{G}_{a,b}$ corresponded to the eigenvalues $\pm 1 : \mathbf{v} \to \mathbf{v}'$

Eigenvectors of $\mathbf{S}_{(4)}$

→ Eigenvectors of $\mathbf{G}_{a,b}$ corresponded to the eigenvalues $\pm i : \mathbf{v} \to \mathbf{v}''$,

which determine a two-sided path by which all the eigen-structure points can be converted between the pair $(\mathbf{C}_{(4)}, \mathbf{S}_{(4)})$ and $\mathbf{G}_{a,b}$.

6. Numerical stability

It is important to check the numerical stability of the proposed bases. Here we will calculate the conditional number of the eigenvector matrix $\mathbf{V}$ for even and odd order n in $\mathbf{S}_{(4)}$ and $\mathbf{C}_{(4)}$ cases. These conditional numbers can serve as an estimate of the numerical stability (and accuracy) when $\mathbf{V}$ and $\mathbf{V}^{-1}$ are calculated numerically.

Conditional numbers for $\mathbf{S}_{(4)}$ are presented by dots in Fig. 1 for even order n (left) and for odd order n (right), for $4 \le n \le 100$. From Fig. 1 we can conclude that the conditional number increases linearly with n in the even n case, and that conditional numbers for odd n are much lower, with, approximately, a logarithmic dependence. By using a heuristic approach we have found approximate formulas for the conditional number in both cases

$$\text{cond}\,\mathbf{V} \approx \begin{cases} 2.5461n & \text{for even } n \\ 0.3374\log(n+4) + 1.9493 & \text{for odd } n. \end{cases} \tag{12}$$

The approximations are presented with the solid lines in Fig. 1. The same results are obtained in the $\mathbf{C}_{(4)}$ case. We can conclude that the odd n case provides excellent numerical stability with conditional number of the eigenvectors matrix very close to 1.

The calculation complexity is n for an odd case and $n^2 + n$ for an even case. This is significantly better when compared with the numerical approach (LAPACK's eig function), where the complexity is $O(n^3)$. It's important to note that the GDFT approach provides an approximate description (including double infinite sums) with a calculation complexity higher than n^2.

7. Conclusion

In this paper, we have proposed a method to calculate eigenvectors of the DST and DCT of type IV. A closed form of the eigenvectors is obtained in both cases. Next we have shown that these eigenvectors can be used to obtain the eigenvectors of the offset DFT with $a = b = -\frac{1}{2}$. Detailed proofs are provided along with a numerical stability test. This paper proposes that instead of focusing on $\mathbf{G}_{a,b}$, the analysis of the eigen-structures of its real and imaginary parts is probably more helpful, since the $\mathbf{C}_{(4)}$ and $\mathbf{S}_{(4)}$ are related to the real and imaginary parts of $\mathbf{G}_{a,b}$. Based on this key point, we have an immediate access to the eigenvectors.

Similar to the DST and DCT of type IV, it becomes possible to identify a set of eigenvectors that could potentially serve as an eigenbasis for any other transform (particularly normalized DTT) which are square root of identity matrix. However, it's important to note that the proposed approach is specifically applicable to these two transforms only.

For future studies, one might explore the identification and modification of columns to achieve a suitable selection for establishing the eigenbasis.

8. Appendix

8.1. Proof of relation (6)

Denote by $X(k,l)$, $k,l = 0,1,\dots,\frac{n-1}{2}$, elements of $\mathbf{S}_o^2$. We have

$$\begin{aligned} X(k,l) &= \sum_{m=0}^{\frac{n-1}{2}} \sin\frac{(4k+1)(4m+1)\pi}{4n}\sin\frac{(4m+1)(4l+1)\pi}{4n} \\ &= \frac{1}{2}\sum_{m=0}^{\frac{n-1}{2}}\cos\frac{(4m+1)(2k-2l)\pi}{2n} - \frac{1}{2}\sum_{m=0}^{\frac{n-1}{2}}\cos\frac{(4m+1)(2k+2l+1)\pi}{2n}. \end{aligned} \tag{13}$$

It is easy to check that for an integer q holds

$$\sum_{m=0}^{\frac{n-1}{2}}\cos\frac{(4m+1)q\pi}{2n}$$

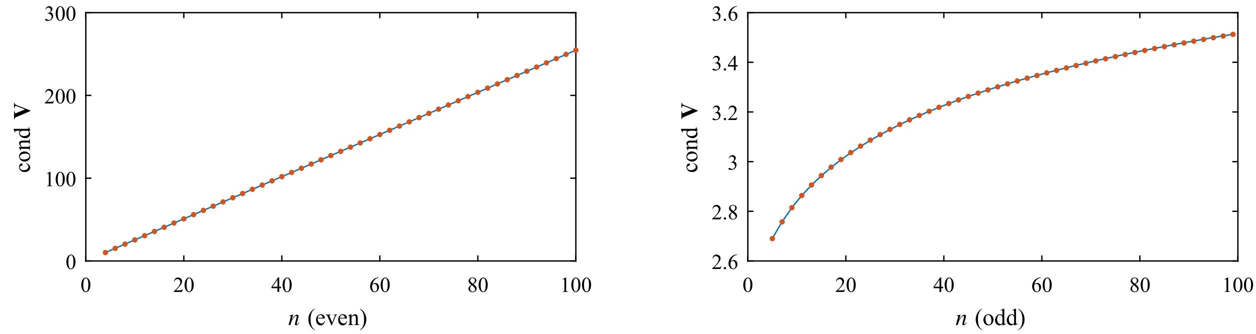

Fig. 1. Conditional numbers of eigenvector matrix $\mathbf{V}$ for even n (left) and for odd n (right). Dots represent actual conditional numbers and line is empirically obtained approximation.

$$= \begin{cases} \frac{n+1}{2} & \text{for } q \text{ divisible by } 4n \\ -\frac{n+1}{2} & \text{for } q \text{ divisible by } 2n \text{ and not divisible by } 4n \\ 0 & \text{for odd } q \\ \frac{1}{2}\sec\frac{q\pi}{2n} & \text{for even } q \text{ not divisible by } 2n. \end{cases}$$

The second sum in (13) is zero since $q = 2k + 2k + 1$ is odd. For the first sum in (13) we have $q = 2(k - l)$ and it is divisible by $2n$ only for $k = l$, resulting in

$$X(k,l) = \begin{cases} \frac{n+1}{4} & \text{for } k = l \\ \frac{1}{4}\sec\frac{(k-l)\pi}{n} & \text{for } k \neq l. \end{cases}$$

This proves (6).

8.2. Proof of Lemma 4

We first follow the proof for the transform $\mathbf{S}_{(4)}$. It is done in three steps.

Step 1. First, we will show that

$$\mathbf{G} = n\mathbf{I}_{n/2} + \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{U}\mathbf{S}_{(4)}\mathbf{Q}.$$

Let us introduce an $n \times n$ matrix $\mathbf{U}$ as

$$\mathbf{U} = \overbrace{\mathbf{J}_2 \oplus \mathbf{J}_2 \oplus \cdots \oplus \mathbf{J}_2}^{n/2 \text{ times}},$$

where $\mathbf{J}_2 =$ is a 2×2 anti-diagonal matrix with ones at the anti-diagonal, and $\oplus$ is a matrix direct sum.

The matrix $\mathbf{G}$ is, by definition,

$$\mathbf{G} = \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{Q}\mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{Q}.$$

Having in mind that $\mathbf{Q}\mathbf{Q}^t = \mathbf{I}_n + \mathbf{U}$ and $\mathbf{Q}^t\mathbf{Q} = 2\mathbf{I}_{n/2}$, we get

$$\mathbf{G} = \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{I}_n\mathbf{S}_{(4)}\mathbf{Q} + \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{U}\mathbf{S}_{(4)}\mathbf{Q} = \frac{n}{2}\mathbf{Q}^t\mathbf{I}_n\mathbf{Q} + \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{U}\mathbf{S}_{(4)}\mathbf{Q}$$
$$= n\mathbf{I}_{n/2} + \mathbf{Q}^t\mathbf{S}_{(4)}\mathbf{U}\mathbf{S}_{(4)}\mathbf{Q}.$$

Step 2. The matrix $\mathbf{G}$ is an X-shaped matrix whose entries are given as follows:

$$G(k,l) = \begin{cases} n + n\cos\frac{\pi}{2n}\cos\frac{(2k+1)\pi}{n} & k = l \text{ and } k + l \neq \frac{n}{2} - 1 \\ n\cos\frac{\pi}{2n}\sin\frac{(2k+1)\pi}{n} & k + l = \frac{n}{2} - 1 \text{ and } k \neq l \\ n + n\cos\frac{\pi}{2n} & k = l \text{ and } k + l = \frac{n}{2} - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Note that the first case is for the diagonal elements of $\mathbf{G}$, the second case stands for the anti-diagonal elements, and the third case is for the central element if the order of matrix $\mathbf{G}$ is odd.

Consider now $n \times n$ matrices $\mathbf{H} = \mathbf{U}\mathbf{S}_{(4)}$ and $\mathbf{W} = \mathbf{S}_{(4)}\mathbf{U}\mathbf{S}_{(4)} = \mathbf{S}_{(4)}\mathbf{H}$. Denote by $S(k,l)$, $U(k,l)$, $H(k,l)$, and $W(k,l)$ the elements of $\mathbf{S}_{(4)}$ $\mathbf{U}$, $\mathbf{H}$, and $\mathbf{W}$, respectively. We have that:

$$U(k,l) = \delta_{k+(-1)^k,l}$$

$$H(k,l) = \sum_{m=0}^{n-1}\delta_{k+(-1)^k,m}S(m,l) = S(k + (-1)^k, l)$$

$$W(k,l) = \sum_{m=0}^{n-1}S(k,m)H(m,k) = \sum_{m=0}^{n-1}S(k,m)S(m + (-1)^m, l).$$

Since $W(k,l) = W(l,k)$ and $S(k,l) = S(l,k)$ we can write

$$W(k,l) = \frac{1}{2}\sum_{m=0}^{n-1}S(k,m)S(m + (-1)^m, l) + S(k, m + (-1)^m)S(m,l). \tag{15}$$

Expanding the product of sine in $S(k,m)S(m + (-1)^m, l)$ and $S(k, m + (-1)^m)S(m,l)$ we get

$$\frac{1}{2}\cos\frac{((2m+1)(k-l) - (2l+1)(-1)^m)\pi}{2n}$$
$$-\frac{1}{2}\cos\frac{((2m+1)(k+l+1) + (2l+1)(-1)^m)\pi}{2n}$$
$$+\frac{1}{2}\cos\frac{((2m+1)(k-l) + (2k+1)(-1)^m)\pi}{2n}$$
$$-\frac{1}{2}\cos\frac{((2m+1)(k+l+1) + (2k+1)(-1)^m)\pi}{2n}.$$

Now we will combine the first and the third term, and the second and the fourth term. We obtain the summand in (15) as

$$\cos\frac{(k+l+1)\pi}{2n}\cos\frac{(2m+1+(-1)^m)(k-l)\pi}{2n}$$
$$-\cos\frac{(k-l)\pi}{2n}\cos\frac{(2m+1+(-1)^m)(k+l+1)\pi}{2n}.$$

Consider now a sum of the form

$$\sum_{m=0}^{n-1}\cos\frac{(2m+1+(-1)^m)q\pi}{2n} = \sum_{m=0}^{n/2-1}\cos\frac{(4m+2)q\pi}{2n} + \sum_{m=0}^{n/2-1}\cos\frac{(4m+2)q\pi}{2n}$$
$$= 2\sum_{m=0}^{n/2-1}\cos\frac{(2m+1)q\pi}{n},$$

where q is an integer. The sum on the right-hand side is equal to zero if q is not divisible by n. When $q = rn$, where r is an integer, the result is $n/2$ for an even r and $-n/2$ for an odd r.

Now we will apply this result to the summations in (15). We will use $q = k - l$ (it is divisible by n only when $k = l$) and $q = k + l + 1$ (that is divisible by n only for $k + l = n - 1$). We obtain

$$W(k,l) = \begin{cases} \frac{n}{2}\cos\frac{(2k+1)\pi}{2n} & \text{for } k = l \\ \frac{n}{2}\cos\frac{(2k+1-n)\pi}{2n} & \text{for } k + l = n - 1 \\ 0 & \text{otherwise,} \end{cases}$$

meaning that $\mathbf{W}$ is an X-shaped matrix.

Now we will find the elements $G(k,l)$ of matrix $\mathbf{G} = n\mathbf{I}_{n/2} + \mathbf{Q}^t\mathbf{W}\mathbf{Q}$. For $k = l$ and $k + l \neq \frac{n}{2} - 1$ we obtain

$$G(k,k) = n + W(2k,2k) + W(2k+1,2k+1) \\ = n + n\cos\frac{\pi}{2n}\cos\frac{(2k+1)\pi}{n}.$$

For $k + l = \frac{n}{2} - 1$ and $k \neq l$ we obtain

$$G(k,\tfrac{n}{2}-1-k) = W(2k,n-1-2k) + W(2k+1,n-1-2k+1) \\ = n\cos\frac{\pi}{2n}\sin\frac{(2k+1)\pi}{n}.$$

In the special case when $k = l$ and $k + l = \frac{n}{2} - 1$ we have that the central element of $\mathbf{G}$, obtained for $k = \frac{n}{4} - \frac{1}{2}$ is

$$G(k,k) = n + W(2k,2k) + W(2k+1,2k+1) + W(2k,2k+1) \\ + W(2k+1,2k) \\ = n + n\cos\frac{\pi}{2n}.$$

The elements $G(k,l)$, when $k \neq l$ or $k + l \neq \frac{n}{2} - 1$, are equal to zero.

Step 3. Based on the previous two steps, we get the desired result. Let us consider a 2×2 principal submatrix of $\mathbf{G}$,

$$\mathbf{P}_m = \begin{bmatrix} G(m,m) & G(m,\frac{n}{2}-1-m) \\ G(\frac{n}{2}-1-m,m) & G(\frac{n}{2}-1-m,\frac{n}{2}-1-m) \end{bmatrix}.$$

One may check that

$$\mathrm{Eigen}(\mathbf{G}) = \bigcup_m \mathrm{Eigen}(\mathbf{P}_m).$$

We have that

$$\mathbf{P}_m = \begin{bmatrix} n + n\cos\frac{\pi}{2n}\cos\frac{(2m+1)\pi}{n} & n\cos\frac{\pi}{2n}\sin\frac{(2m+1)\pi}{n} \\ n\cos\frac{\pi}{2n}\sin\frac{(2m+1)\pi}{n} & n - n\cos\frac{\pi}{2n}\cos\frac{(2*m+1)\pi}{n} \end{bmatrix},$$

with the eigenvalues

$$\lambda_{1,2} = n \pm n\cos\frac{\pi}{2n},$$

independent of m, meaning that $\mathrm{Eigen}(\mathbf{G}) = \{n + n\cos\frac{\pi}{2n}, n - n\cos\frac{\pi}{2n}\}$. This completes the proof of Lemma 4 in the $\mathbf{T} = \mathbf{S}_{(4)}$ case.

In a similar way, one may get Lemma 4 for $\mathbf{T} = \mathbf{C}_{(4)}$. Only the formula of the matrix $\mathbf{G}$ is changed as follows:

$$G(k,l) = \begin{cases} n + n\cos\frac{\pi}{2n}\cos\frac{(2k+1)\pi}{n} & k = l, k + l \neq \frac{n}{2} - 1 \\ -n\cos\frac{\pi}{2n}\sin\frac{(2k+1)\pi}{n} & k \neq l, k + l = \frac{n}{2} - 1 \\ n - n\cos\frac{\pi}{2n} & k = l, k + l = \frac{n}{2} - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

CRediT authorship contribution statement

Ali Bagheri Bardi: Conceptualization, Formal analysis, Methodology, Writing – original draft. **Miloš Daković:** Conceptualization, Formal analysis, Methodology, Software, Validation, Writing – original draft. **Taher Yazdanpanah:** Formal analysis, Methodology, Writing – review & editing. **Fatemeh Zarei:** Formal analysis, Methodology, Writing – review & editing. **Ljubiša Stanković:** Formal analysis, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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