



Eigenvalues of symmetric non-normalized discrete trigonometric transforms

Ali Bagheri Bardi ^{a,1}, Miloš Daković ^b, Taher Yazdanpanah ^a, Ljubiša Stanković ^{b,*}

^a Persian Gulf University, Busher, Iran

^b University of Montenegro, Podgorica, Montenegro

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ABSTRACT

A comprehensive approach to the spectrum characterization (derivation of eigenvalues and the corresponding multiplicities) for non-normalized, symmetric discrete trigonometric transforms (DTT) is presented in the paper. Eight types of the DTT are analyzed. New explicit analytic expressions for the eigenvalues, together with their multiplicities, for the cases of three DTT ($DCT_{(1)}$, $DCT_{(5)}$, and $DST_{(8)}$), are the main contribution of this paper. Moreover, the presented theory is supplemented by new, original derivations for the closed-form expressions of the square and the trace of all symmetric DTT matrices including existing results and the new ones.

1. Introduction

Discrete Trigonometric Transforms (DTT) are irreplaceable tools in signal and image processing applications. There exist 16 types of the DTT [1–9] divided into two classes: Discrete Cosine Transforms (DCT) and Discrete Sine Transforms (DST). In each class, eight types of these transforms are defined. In addition, there are non-normalized and normalized variants of the DTT. All of these transforms are linear, and therefore, for a given signal of length n , they can be suitably represented using $n \times n$ transformation matrices.

Herein, we will focus on the symmetric non-normalized DTT, that is, on the DST of type 1, 4, 5, and 8 and the DCT of type 1, 4, 5, and 8. The elements of the transformation matrix for each analyzed DTT are given in Table 1, where $k = 0, 1, \dots, n-1$ is a row index and $l = 0, 1, \dots, n-1$ is a column index. The DTT of type m are denoted as $DCT_{(m)}$ and $DST_{(m)}$, and the corresponding transformation matrices are denoted as $C_{(m)}$ and $S_{(m)}$.

The eigenvalues of a symmetric transform indicate how much of the signal's energy is concentrated in different frequency components. This information is important in data compression [10]. Eigenvalues are also related to the orthogonality of basis. Orthogonal bases simplify signal representation and manipulation. In image and signal analysis, this can lead to better feature extraction and noise reduction [11,12]. The multiplicities of eigenvalues provide insights into the structure of the transformed signal. High multiplicities suggest repeated or symmetrical patterns in the data. This knowledge is valuable for tasks like pattern recognition [13].

Eigenvalues of DTTs are studied in [14–20]. However, the discussion is limited to five types of symmetric normalized DTT (DCT of

types 4 and 8 and DST of types 1, 4, and 5) where the square of the transformation matrix A equals the identity matrix, $A^2 = I$.

Eigendecompositions of the normalized $DST_{(4)}$ and the $DCT_{(4)}$ are analyzed in [14]. Therein, the authors use the Generalized Discrete Fourier Transform (GDFT), and the theory of commuting matrices in order to obtain approximate eigendecompositions of $DST_{(4)}$ and $DCT_{(4)}$. In [15] it has been shown that the normalized $DCT_{(1)}$ and $DST_{(1)}$ eigenvectors can be attained from the DFT eigenvectors. The offset Discrete Fourier Transform (DFT) is used in [16], where it has been shown that an even-order normalized $DCT_{(4)}$, $DST_{(4)}$, and $DST_{(8)}$ can be viewed as a special case of an even-order offset DFT. This approach has led to the eigenvalues (and their corresponding multiplicities) for these three types of DTT. The approach based on commuting matrices is used in [18,20] to determine the eigenvectors of some DTT. Non-symmetric DTT are briefly analyzed in [19], providing a conjecture that all eigenvalues are distinct for non-symmetric DTT of arbitrary order.

Our aim is to find the eigenvalues, with their corresponding multiplicities, in an analytic way, for each considered non-normalized symmetric type of DTT. Applying some well-known trigonometric identities, we directly obtain the square and the trace of all eight types of DTT matrices. We observe that the square of the transformation matrix for three types of non-normalized DTT, (the $DCT_{(1)}$, $DCT_{(5)}$, and $DST_{(8)}$) is not a multiple of the identity matrix that makes the discussion quite challenging. We develop a unified analytic approach to non-normalized symmetric DTT eigenvalues (and corresponding multiplicities), containing novel results for the $DCT_{(1)}$, $DCT_{(5)}$, and $DST_{(8)}$. Mathematically relevant expressions for the square and the trace of the analyzed DTT

* Corresponding author.

E-mail addresses: bagheri@pgu.ac.ir (A.B. Bardi), milos@ucg.ac.me (M. Daković), yazdanpanah@pgu.ac.ir (T. Yazdanpanah), ljubisa@ucg.ac.me (L. Stanković).

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Table 1
Definitions for symmetric, non-normalized DTT.

Type	$(s_{kl})_{k,l=0}^{n-1}$	Type	$(c_{kl})_{k,l=0}^{n-1}$
DST ₍₁₎	$\sin \frac{(k+1)(l+1)\pi}{n+1}$	DCT ₍₁₎	$\cos \frac{kl\pi}{n-1}$
DST ₍₄₎	$\sin \frac{(2k+1)(2l+1)\pi}{4n}$	DCT ₍₄₎	$\cos \frac{(2k+1)(2l+1)\pi}{4n}$
DST ₍₅₎	$\sin \frac{2(k+1)(l+1)\pi}{2n+1}$	DCT ₍₅₎	$\cos \frac{2kl\pi}{2n-1}$
DST ₍₈₎	$\sin \frac{(2k+1)(2l+1)\pi}{4n-2}$	DCT ₍₈₎	$\cos \frac{(2k+1)(2l+1)\pi}{4n+2}$

Note that DCT₍₁₎ is not defined for $n = 1$.

matrices arise as intermediate results, that are used for the calculation of the eigenvalues and their corresponding multiplicities.

Both normalized and non-normalized versions of DTTs are categorized into symmetric and non-symmetric types. The discussion of the eigenvalue problem for symmetric types is entirely different from that for non-symmetric types. In the case of symmetric types, excluding DCT of types 1 and 4 and DST of type 8, normalized and non-normalized symmetric DTTs are the same up to some scaling factor. Therefore, to facilitate a parallel discussion on determining the eigenvalues of the normalized symmetric DTTs, we also need to address DCT of types 1 and 4 and DST of type 8. Trigonometric identities frequently applied in the text allow for direct verification that these matrices are square roots of the identity matrix, thus resulting in eigenvalues of ± 1 . Furthermore, by employing a similar process as used in this paper, one can compute the trace to determine the multiplicities.

The main results are summarized in Section 2. Squares of DTT matrices are evaluated in Section 3 while their traces are derived in Section 4. In Section 5, the analytic proofs for the results presented in Section 2 are presented.

2. Results

The eigenvalues along with their corresponding multiplicities, for the considered types of DTT are presented in Tables 2 and 3. These expressions are the main result of this paper. Here we assume $n \geq 2$ for DCT₍₄₎, DST₍₄₎, DCT₍₈₎, DST₍₅₎, and DST₍₁₎ case, and $n \geq 4$ for DCT₍₅₎ and DST₍₈₎. For DCT₍₁₎ we assume $n \geq 6$.

By carefully observing these forms, we can see that five DTT types have only two distinct eigenvalues and that for an odd n , in each case, the multiplicity of the positive eigenvalue is greater by one than the multiplicity of the corresponding negative eigenvalue.

For the DCT of type 5 and the DST of type 8 there are four eigenvalues. Two of them have multiplicity one, and the multiplicities of other two eigenvalues are equal in the case of even n , or differ by one in the case of odd n , where again the positive eigenvalue has greater multiplicity.

The most complicated case is the DCT of type 1, where there are six distinct eigenvalues. Four of them have multiplicity one and the remaining two eigenvalues have equal multiplicities (for even n), whereas the positive eigenvalue multiplicity is greater by one than the negative eigenvalue multiplicity (for odd n case).

In following sections, we provide detailed discussion, derivations and proofs for the presented results.

3. Square of transformation matrix

In order to compute entries t_{kl} of the square of the DTT matrices in Table 1, we need to apply some well-known trigonometric identities [21, p. 37].

Lagrange's trigonometric identity states that for $\theta \neq 2k\pi$, where k is an integer, holds

$$\sum_{m=0}^n \cos m\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}}.$$

Table 2
Eigenvalues and corresponding multiplicities.

DTT type	Eigenvalue	Multiplicity	
		Even n	Odd n
DCT ₍₄₎ , DST ₍₄₎	$\lambda_1 = -\sqrt{\frac{n}{2}}$	$\frac{n}{2}$	$\frac{n-1}{2}$
	$\lambda_2 = \sqrt{\frac{n}{2}}$	$\frac{n}{2}$	$\frac{n+1}{2}$
DCT ₍₈₎ , DST ₍₅₎	$\lambda_1 = -\sqrt{\frac{2n+1}{4}}$	$\frac{n}{2}$	$\frac{n-1}{2}$
	$\lambda_2 = \sqrt{\frac{2n+1}{4}}$	$\frac{n}{2}$	$\frac{n+1}{2}$
DST ₍₁₎	$\lambda_1 = -\sqrt{\frac{n+1}{2}}$	$\frac{n}{2}$	$\frac{n-1}{2}$
	$\lambda_2 = \sqrt{\frac{n+1}{2}}$	$\frac{n}{2}$	$\frac{n+1}{2}$
DCT ₍₅₎	$\lambda_1 = \frac{1}{4} - \sqrt{n - \frac{7}{16}}$	1	1
	$\lambda_2 = -\sqrt{\frac{2n-1}{4}}$	$\frac{n}{2} - 1$	$\frac{n-3}{2}$
	$\lambda_3 = \sqrt{\frac{2n-1}{4}}$	$\frac{n}{2} - 1$	$\frac{n-1}{2}$
	$\lambda_4 = \frac{1}{4} + \sqrt{n - \frac{7}{16}}$	1	1
DST ₍₈₎	$\lambda_1 = -\frac{(-1)^n}{4} - \sqrt{n - \frac{7}{16}}$	1	1
	$\lambda_2 = -\sqrt{\frac{2n-1}{4}}$	$\frac{n}{2} - 1$	$\frac{n-3}{2}$
	$\lambda_3 = \sqrt{\frac{2n-1}{4}}$	$\frac{n}{2} - 1$	$\frac{n-1}{2}$
	$\lambda_4 = -\frac{(-1)^n}{4} + \sqrt{n - \frac{7}{16}}$	1	1

Table 3
DCT type 1 eigenvalues.

Even n		Odd n	
Eigenvalue	Mult.	Eigenvalue	Mult.
$\lambda_1 = -\frac{\sqrt{2}}{4} - \sqrt{n - \frac{7}{8}}$	1	$\lambda_1 = -\sqrt{n-1}$	1
$\lambda_2 = \frac{\sqrt{2}}{4} - \sqrt{n - \frac{7}{8}}$	1	$\lambda_2 = \frac{1}{2} - \sqrt{n - \frac{3}{4}}$	1
$\lambda_3 = -\sqrt{\frac{n-1}{2}}$	$\frac{n}{2} - 2$	$\lambda_3 = -\sqrt{\frac{n-1}{2}}$	$\frac{n-5}{2}$
$\lambda_4 = \sqrt{\frac{n-1}{2}}$	$\frac{n}{2} - 2$	$\lambda_4 = \sqrt{\frac{n-1}{2}}$	$\frac{n-3}{2}$
$\lambda_5 = -\frac{\sqrt{2}}{4} + \sqrt{n - \frac{7}{8}}$	1	$\lambda_5 = \sqrt{n-1}$	1
$\lambda_6 = \frac{\sqrt{2}}{4} + \sqrt{n - \frac{7}{8}}$	1	$\lambda_6 = \frac{1}{2} + \sqrt{n - \frac{3}{4}}$	1

Table 4
Elements t_{kl} of squared transformation matrix.

Type	t_{kl} , for $k \neq l$	t_{kl} , for $k = l$	Type	t_{kl} , for $k \neq l$	t_{kl} , for $k = l$
DST ₍₁₎	0	$\frac{n+1}{2}$	DCT ₍₁₎	$\frac{1+(-1)^{k+l}}{2}$	$\frac{n+1}{2} + \frac{n-1}{2}(\delta_{k,0} + \delta_{k,n-1})$
DST ₍₄₎	0	$\frac{n}{2}$	DCT ₍₄₎	0	$\frac{n}{2}$
DST ₍₅₎	0	$\frac{2n+1}{4}$	DCT ₍₅₎	$\frac{1}{2}$	$\frac{2n+1}{4} + \frac{2n-1}{4}\delta_{k,0}$
DST ₍₈₎	$\frac{(-1)^{k+l}}{2}$	$\frac{2n+1}{4} + \frac{2n-1}{4}\delta_{k,n-1}$	DCT ₍₈₎	0	$\frac{2n+1}{4}$

$\delta_{k,m}$ is Kronecker delta, $\delta_{k,m} = 0$ for $k \neq m$ and $\delta_{k,m} = 1$ for $k = m$.

Using this identity for $\theta = (a\pi)/n$, where a is an integer that is not divisible with $2n$, and for $\theta = (2b\pi)/(2n+1)$, where b is an integer not divisible with $2n+1$, we get the following two identities

$$\sum_{m=0}^n \cos \frac{m a \pi}{n} = \frac{1 + (-1)^a}{2} = \begin{cases} 1 & \text{for even } a \\ 0 & \text{for odd } a \end{cases} \quad (1)$$

$$\sum_{m=0}^n \cos \frac{2mb\pi}{2n+1} = \frac{1}{2}. \quad (2)$$

For an integer a not divisible by $2n$, the following identity holds:

$$\sum_{m=0}^{n-1} \cos \frac{(2m+1)a\pi}{2n} = 0. \quad (3)$$

For an integer a not divisible by $2n+1$,

$$\sum_{m=0}^{n-1} \cos \frac{(2m+1)a\pi}{2n+1} = \frac{(-1)^{a+1}}{2} \quad (4)$$

holds, while for an integer a not divisible by $2n+2$, we have

$$\sum_{m=1}^n \cos \frac{ma\pi}{n+1} = -\frac{(-1)^a + 1}{2} = \begin{cases} -1 & \text{for even } a \\ 0 & \text{for odd } a. \end{cases} \quad (5)$$

Furthermore, for an integer a not divisible by $2n+1$, we have the following:

$$\sum_{m=0}^{n-1} \cos \frac{2(m+1)a\pi}{2n+1} = -\frac{1}{2}. \quad (6)$$

Elements of the squared transformation matrix, are derived in the sequel, for each analyzed DTT. The results are summarized in Table 4.

3.1. DCT type 1 case

Elements of the squared transformation matrix are

$$t_{kl} = \sum_{m=0}^{n-1} \cos \frac{km\pi}{n-1} \cos \frac{ml\pi}{n-1} = \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{m(k+l)\pi}{n-1} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{m(k-l)\pi}{n-1}.$$

Using (1), for $k \neq l$, we further get

$$t_{kl} = \frac{1 + (-1)^{k+l}}{2} = \begin{cases} 1 & \text{for even } k+l \\ 0 & \text{for odd } k+l. \end{cases}$$

For $k = l$, we obtain

$$t_{kk} = \begin{cases} \frac{n+1}{2} & \text{for } k = 1, 2, \dots, n-2 \\ n & \text{for } k = 0 \text{ or } k = n-1. \end{cases}$$

3.2. DCT type 4 case

In this case, the elements of the squared transformation matrix are

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \cos \frac{(2k+1)(2m+1)\pi}{4n} \cos \frac{(2m+1)(2l+1)\pi}{4n} \\ &= \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k+l+1)\pi}{2n} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k-l)\pi}{2n}. \end{aligned}$$

For $k \neq l$, using (3), we get $t_{kl} = 0$. For $k = l$, using (3) for the first sum only we have

$$t_{kk} = \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(2k+1)\pi}{2n} + \frac{n}{2} = \frac{n}{2}.$$

3.3. DCT type 5 case

For this type of the transform we have

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \cos \frac{2km\pi}{2n-1} \cos \frac{2ml\pi}{2n-1} = \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{2m(k+l)\pi}{2n-1} \\ &+ \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{2m(k-l)\pi}{2n-1}. \end{aligned}$$

For $k \neq l$ using (2) we obtain $t_{kl} = 1/2$, while for $k = l$ we get

$$t_{kk} = \begin{cases} \frac{2n+1}{4} & \text{for } k \neq 0 \\ n & \text{for } k = 0. \end{cases}$$

3.4. DCT type 8 case

The squared transformation matrix elements for the DCT₍₈₎ are given by

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \cos \frac{(2k+1)(2m+1)\pi}{4n+2} \cos \frac{(2m+1)(2l+1)\pi}{4n+2} \\ &= \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k+l+1)\pi}{2n+1} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k-l)\pi}{2n+1}. \end{aligned}$$

For $k \neq l$, using (4) we get $t_{kl} = 0$. Otherwise, for $k = l$, using (4) for the first sum, we get

$$t_{kk} = \frac{1}{4} + \frac{n}{2} = \frac{2n+1}{4}.$$

3.5. DST type 1 case

In the case of DST₍₁₎, the elements of the transformation matrix are given by

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \sin \frac{(k+1)(m+1)\pi}{n+1} \sin \frac{(m+1)(l+1)\pi}{n+1} \\ &= -\frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(m+1)(k+l+2)\pi}{n+1} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(m+1)(k-l)\pi}{n+1}. \end{aligned}$$

For $k \neq l$, using (5), we have $t_{kl} = 0$, while for $k = l$, $t_{kk} = (n+1)/2$ holds.

3.6. DST type 4 case

The elements of the squared DST₍₄₎ transformation matrix are

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \sin \frac{(2k+1)(2m+1)\pi}{4n} \sin \frac{(2m+1)(2l+1)\pi}{4n} \\ &= -\frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k+l+1)\pi}{2n} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k-l)\pi}{2n}. \end{aligned}$$

For $k \neq l$, using (3) we have $t_{kl} = 0$, and for $k = l$ we get $t_{kk} = n/2$.

3.7. DST type 5 case

Next, we consider the elements of the squared DST₍₅₎ transformation matrix. These elements are given by

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \sin \frac{2(k+1)(m+1)\pi}{2n+1} \sin \frac{2(m+1)(l+1)\pi}{2n+1} \\ &= -\frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{2(m+1)(k+l+2)\pi}{2n+1} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{2(m+1)(k-l)\pi}{2n+1}. \end{aligned}$$

For $k \neq l$, using (6) we have $t_{kl} = 0$, and for $k = l$ we get $t_{kk} = (2n+1)/4$.

3.8. DST type 8 case

For the DST₍₈₎ the elements of squared matrix are

$$\begin{aligned} t_{kl} &= \sum_{m=0}^{n-1} \sin \frac{(2k+1)(2m+1)\pi}{4n-2} \sin \frac{(2m+1)(2l+1)\pi}{4n-2} \\ &= -\frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k+l+1)\pi}{2n-1} + \frac{1}{2} \sum_{m=0}^{n-1} \cos \frac{(2m+1)(k-l)\pi}{2n-1}. \end{aligned}$$

In this case, for $k \neq l$, using (4) we get $t_{kl} = (-1)^{k+l}/2$, whereas for $k = l$ we have

$$t_{kk} = \begin{cases} \frac{2n+1}{4} & \text{for } k = 0, 1, \dots, n-2 \\ n & \text{for } k = n-1. \end{cases}$$

Table 5
Traces of DTT matrices.

Type	Trace		Type	Trace	
	Even n	Odd n		Even n	Odd n
DST ₍₁₎	0	$\sqrt{\frac{n+1}{2}}$	DCT ₍₁₎	0	$\frac{2+\sqrt{2n-2}}{2}$
DST ₍₄₎	0	$\sqrt{\frac{n}{2}}$	DCT ₍₄₎	0	$\sqrt{\frac{n}{2}}$
DST ₍₅₎	0	$\frac{\sqrt{2n+1}}{2}$	DCT ₍₅₎	$\frac{1}{2}$	$\frac{1+\sqrt{2n-1}}{2}$
DST ₍₈₎	$-\frac{1}{2}$	$\frac{1+\sqrt{2n-1}}{2}$	DCT ₍₈₎	0	$\frac{\sqrt{2n+1}}{2}$

4. Traces of the transformation matrices

In this section, we will determine the trace of the transformation matrices in Table 1. The results are summarized in Table 5. To this aim, we use the following identities [21, p. 37]

$$\sum_{k=0}^{m-1} \cos \frac{2k^2\pi}{m} = \frac{\sqrt{m}}{2} \left(1 + \cos \frac{m\pi}{2} + \sin \frac{m\pi}{2} \right) \quad (7)$$

$$\sum_{k=0}^{m-1} \sin \frac{2k^2\pi}{m} = \frac{\sqrt{m}}{2} \left(1 + \cos \frac{m\pi}{2} - \sin \frac{m\pi}{2} \right). \quad (8)$$

For the DCT₍₁₎, DCT₍₄₎, DST₍₁₎, and DST₍₄₎, for an even number n , the diagonal elements, d_k , of transformation matrix are anti-symmetric, that is, $d_k = -d_{n-1-k}$. The trace of these matrices is obviously zero

$$\text{Tr } \mathbf{C}_{(1)} = \text{Tr } \mathbf{C}_{(4)} = \text{Tr } \mathbf{S}_{(1)} = \text{Tr } \mathbf{S}_{(4)} = 0, \text{ for even } n. \quad (9)$$

Next, we derive the value of the transformation matrix trace in other cases.

4.1. DCT type 1, odd n case

From (7), using $m = 2(n-1)$, we get

$$\begin{aligned} \sqrt{2(n-1)} &= \sum_{k=0}^{2n-3} \cos \frac{k^2\pi}{n-1} = \sum_{k=0}^{n-2} \cos \frac{k^2\pi}{n-1} + \sum_{k=0}^{n-2} \cos \frac{(k+n-1)^2\pi}{n-1} \\ &= 2 \sum_{k=0}^{n-2} \cos \frac{k^2\pi}{n-1} = 2 \sum_{k=0}^{n-1} \cos \frac{k^2\pi}{n-1} - 2. \end{aligned}$$

Previous result is the basis for the explicit expression for the trace of the DCT type 1, $\mathbf{C}_{(1)}$, given as follows

$$\text{Tr } \mathbf{C}_{(1)} = \sum_{k=0}^{n-1} \cos \frac{k^2\pi}{n-1} = \begin{cases} 0 & \text{for even } n \\ \frac{2+\sqrt{2n-2}}{2} & \text{for odd } n. \end{cases} \quad (10)$$

4.2. DCT type 4, odd n case

Substituting $m = 8n$ in (7) we get

$$\begin{aligned} 2\sqrt{2n} &= \sum_{k=0}^{8n-1} \cos \frac{k^2\pi}{4n} = \sum_{k=0}^{4n-1} \cos \frac{k^2\pi}{4n} + \sum_{k=0}^{4n-1} \cos \frac{(k+4n)^2\pi}{4n} \\ &= 2 \sum_{k=0}^{4n-1} \cos \frac{k^2\pi}{4n} = 2 \sum_{k=0}^{2n-1} \cos \frac{(2k)^2\pi}{4n} + 2 \sum_{k=0}^{2n-1} \cos \frac{(2k+1)^2\pi}{4n}. \end{aligned}$$

The first sum follows from (7) when we put $m = 2n$, and since n is odd it is equal to zero. Let us decompose the second sum as

$$\begin{aligned} 2\sqrt{2n} &= 2 \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n} + 2 \sum_{k=0}^{n-1} \cos \frac{(2n+2k+1)^2\pi}{4n} \\ &= 4 \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n}. \end{aligned}$$

Now we get the trace for this transformation matrix as

$$\text{Tr } \mathbf{C}_{(4)} = \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n} = \begin{cases} 0 & \text{for even } n \\ \sqrt{\frac{n}{2}} & \text{for odd } n. \end{cases} \quad (11)$$

4.3. DCT type 5 case

For the DCT₍₅₎ we can use (7), with $m = 2n-1$, to obtain

$$\begin{aligned} &\frac{\sqrt{2n-1}}{2} (1 + (-1)^n) \\ &= \sum_{k=0}^{2n-2} \cos \frac{2k^2\pi}{2n-1} = \sum_{k=0}^{n-1} \cos \frac{2k^2\pi}{2n-1} + \sum_{k=0}^{n-2} \cos \frac{2(2n-2-k)^2\pi}{2n-1} \\ &= \sum_{k=0}^{n-1} \cos \frac{2k^2\pi}{2n-1} + \sum_{k=0}^{n-2} \cos \frac{2(k+1)^2\pi}{2n-1} = 2 \sum_{k=0}^{n-1} \cos \frac{2k^2\pi}{2n-1} - 1. \end{aligned}$$

Finally, the trace of $\mathbf{C}_{(5)}$ is given as

$$\text{Tr } \mathbf{C}_{(5)} = \sum_{k=0}^{n-1} \cos \frac{2k^2\pi}{2n-1} = \begin{cases} \frac{1}{2} & \text{for even } n \\ \frac{1+\sqrt{2n-1}}{2} & \text{for odd } n. \end{cases} \quad (12)$$

4.4. DCT type 8 case

For the DCT₍₈₎ we can set $m = 8n+4$ into (7), further leading to

$$\begin{aligned} \sqrt{8n+4} &= \sum_{k=0}^{8n+3} \cos \frac{k^2\pi}{4n+2} = \sum_{k=0}^{4n+1} \cos \frac{k^2\pi}{4n+2} + \sum_{k=0}^{4n+1} \cos \frac{(k+4n+2)^2\pi}{4n+2} \\ &= 2 \sum_{k=0}^{4n+1} \cos \frac{k^2\pi}{4n+2} = 2 \sum_{k=0}^{2n} \cos \frac{2k^2\pi}{2n+1} + 2 \sum_{k=0}^{2n} \cos \frac{(2k+1)^2\pi}{4n+2} \\ &= 2 \sum_{k=0}^{2n} \cos \frac{(2k)^2\pi}{4n+2} + 2 \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n+2} \\ &\quad + 2 \sum_{k=0}^n \cos \frac{(2(k+n)+1)^2\pi}{4n+2}. \end{aligned}$$

The first sum is equal to

$$\sum_{k=0}^{2n} \cos \frac{2k^2\pi}{2n+1} = \frac{\sqrt{2n+1}}{2} (1 + (-1)^n),$$

according to (7) with $m = 2n+1$.

The third sum can be written as

$$\begin{aligned} \sum_{k=0}^n \cos \frac{(2(k+n)+1)^2\pi}{4n+2} &= \sum_{k=0}^n \cos \left(\frac{2k^2\pi}{2n+1} + \frac{2n+1}{2} \pi \right) \\ &= (-1)^{n+1} \sum_{k=0}^n \sin \frac{2k^2\pi}{2n+1}. \end{aligned}$$

From (8), using $m = 2n+1$, we get

$$\begin{aligned} &\frac{\sqrt{2n+1}}{2} (1 - (-1)^n) = \sum_{k=0}^{2n} \sin \frac{2k^2\pi}{2n+1} = \sum_{k=0}^n \sin \frac{2k^2\pi}{2n+1} + \sum_{k=0}^{n-1} \sin \frac{2(2n-k)^2\pi}{2n+1} \\ &= \sum_{k=0}^n \sin \frac{2k^2\pi}{2n+1} + \sum_{k=0}^{n-1} \sin \frac{2(k+1)^2\pi}{2n+1} = 2 \sum_{k=0}^n \sin \frac{2k^2\pi}{2n+1}. \end{aligned}$$

Now we have

$$\sqrt{8n+4} = \sqrt{2n+1} (1 + (-1)^n) + 2 \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n+2} + \frac{\sqrt{2n+1}}{2} (1 - (-1)^n)$$

or

$$\sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n+2} = \frac{\sqrt{2n+1}}{4} (1 - (-1)^n)$$

resulting in the trace of the DCT₍₈₎ of the form

$$\text{Tr } \mathbf{C}_{(8)} = \sum_{k=0}^{n-1} \cos \frac{(2k+1)^2\pi}{4n+2} = \begin{cases} 0 & \text{for even } n \\ \frac{\sqrt{2n+1}}{2} & \text{for odd } n. \end{cases} \quad (13)$$

4.5. DST type 1, odd n case

Using $m = 2(n + 1)$ in (8) we get

$$\begin{aligned} \sqrt{2(n+1)} &= \sum_{k=0}^{2n+1} \sin \frac{k^2 \pi}{n+1} = \sum_{k=0}^{2n} \sin \frac{(k+1)^2 \pi}{n+1} = \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1} \\ &\quad + \sum_{k=0}^n \sin \frac{(n+1+k)^2 \pi}{n+1} \\ &= \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1} + \sum_{k=0}^n \sin \frac{k^2 \pi}{n+1} = \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1} \\ &\quad + \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1} \\ &= 2 \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1}. \end{aligned}$$

The trace of this transformation matrix is then

$$\text{Tr } \mathbf{S}_{(1)} = \sum_{k=0}^{n-1} \sin \frac{(k+1)^2 \pi}{n+1} = \begin{cases} 0 & \text{for even } n \\ \sqrt{\frac{n+1}{2}} & \text{for odd } n. \end{cases} \quad (14)$$

4.6. DST type 4, odd n case

Let start from (8) with $m = 8n$, to get

$$\begin{aligned} 2\sqrt{2n} &= \sum_{k=0}^{8n-1} \sin \frac{k^2 \pi}{4n} = \sum_{k=0}^{4n-1} \sin \frac{k^2 \pi}{4n} + \sum_{k=0}^{4n-1} \sin \frac{(4n+k)^2 \pi}{4n} \\ &= 2 \sum_{k=0}^{4n-1} \sin \frac{k^2 \pi}{4n} = 2 \sum_{k=0}^{2n-1} \sin \frac{(2k)^2 \pi}{4n} + 2 \sum_{k=0}^{2n-1} \sin \frac{(2k+1)^2 \pi}{4n}. \end{aligned}$$

In the last equation, the first sum is zero using (8) with $m = 2n$, and having in mind that n is odd, so we have

$$\begin{aligned} 2\sqrt{2n} &= 2 \sum_{k=0}^{2n-1} \sin \frac{(2k+1)^2 \pi}{4n} = 2 \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n} \\ &\quad + 2 \sum_{k=0}^{n-1} \sin \frac{(2n+2k+1)^2 \pi}{4n} \\ &= 4 \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n}. \end{aligned}$$

Now we get the trace of $\text{DST}_{(4)}$ as

$$\text{Tr } \mathbf{S}_{(4)} = \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n} = \begin{cases} 0 & \text{for even } n \\ \sqrt{\frac{n}{2}} & \text{for odd } n. \end{cases} \quad (15)$$

4.7. DST type 5 case

Using Eq. (8) with $m = 2n + 1$ we can write

$$\begin{aligned} \frac{\sqrt{2n+1}}{2} (1 - (-1)^n) &= \sum_{k=0}^{2n} \sin \frac{2k^2 \pi}{2n+1} = \sum_{k=0}^{2n-1} \sin \frac{2(k+1)^2 \pi}{2n+1} \\ &= \sum_{k=0}^{n-1} \sin \frac{2(k+1)^2 \pi}{2n+1} + \sum_{k=0}^{n-1} \sin \frac{2(2n-k)^2 \pi}{2n+1} \\ &= 2 \sum_{k=0}^{n-1} \sin \frac{2(k+1)^2 \pi}{2n+1}. \end{aligned}$$

The trace for the $\text{DST}_{(5)}$ is now obtained in the following explicit form

$$\text{Tr } \mathbf{S}_{(5)} = \sum_{k=0}^{n-1} \sin \frac{2(k+1)^2 \pi}{2n+1} = \begin{cases} 0 & \text{for even } n \\ \sqrt{\frac{2n+1}{2}} & \text{for odd } n. \end{cases} \quad (16)$$

4.8. DST type 8 case

Eq. (8), with $m = 8n - 4$, can be transformed in the following way

$$\begin{aligned} \sqrt{8n-4} &= \sum_{k=0}^{8n-5} \sin \frac{k^2 \pi}{4n-2} = \sum_{k=0}^{4n-3} \sin \frac{k^2 \pi}{4n-2} + \sum_{k=0}^{4n-3} \sin \frac{(4n-2+k)^2 \pi}{4n-2} \\ &= 2 \sum_{k=0}^{4n-3} \sin \frac{k^2 \pi}{4n-2} = 2 \sum_{k=0}^{2n-2} \sin \frac{2k^2 \pi}{4n-2} + 2 \sum_{k=0}^{2n-2} \sin \frac{(2k+1)^2 \pi}{4n-2} \\ &= 2 \sum_{k=0}^{2n-2} \sin \frac{(2k)^2 \pi}{4n-2} + 2 \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n-2} \\ &\quad + 2 \sum_{k=0}^{n-2} \sin \frac{(2(k+n)+1)^2 \pi}{4n-2} \end{aligned}$$

The first sum can be calculated using (8) with $m = 2n - 1$ as

$$2 \sum_{k=0}^{2n-2} \sin \frac{(2k)^2 \pi}{4n-2} = \sqrt{2n-1} (1 + (-1)^n).$$

The third sum can be written as

$$\begin{aligned} \sum_{k=0}^{n-2} \sin \frac{(2(k+n)+1)^2 \pi}{4n-2} &= \sum_{k=0}^{n-2} \sin \left(\frac{2(k+1)^2 \pi}{2n-1} + \frac{2n-1}{2} \pi \right) \\ &= (-1)^{n+1} \sum_{k=0}^{n-2} \cos \left(\frac{2(k+1)^2 \pi}{2n-1} \right). \end{aligned}$$

Using (7) with $m = 2n - 1$ we get

$$\begin{aligned} \frac{\sqrt{2n-1}}{2} (1 - (-1)^n) &= \sum_{k=0}^{2n-2} \cos \frac{2k^2 \pi}{2n-1} = 1 + \sum_{k=0}^{n-2} \cos \frac{2(k+1)^2 \pi}{2n-1} \\ &\quad + \sum_{k=0}^{n-2} \cos \frac{2(2n-2-k)^2 \pi}{2n-1} \\ &= 1 + 2 \sum_{k=0}^{n-2} \cos \frac{2(k+1)^2 \pi}{2n-1}. \end{aligned}$$

Now, we can write

$$\begin{aligned} \sqrt{8n-4} &= \sqrt{2n-1} (1 + (-1)^n) + 2 \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n+2} \\ &\quad - (-1)^n \left(\frac{\sqrt{2n-1}}{2} (1 - (-1)^n) - 1 \right), \end{aligned}$$

resulting in

$$\sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n+2} = \frac{\sqrt{2n-1}}{4} (1 - (-1)^n) - \frac{(-1)^n}{2},$$

and finally

$$\text{Tr } \mathbf{S}_{(8)} = \sum_{k=0}^{n-1} \sin \frac{(2k+1)^2 \pi}{4n-2} = \begin{cases} -\frac{1}{2} & \text{for even } n \\ \frac{1+\sqrt{2n-1}}{2} & \text{for odd } n. \end{cases} \quad (17)$$

5. Proofs for DTT eigenvalues and their multiplicities

Within this section we will use notation \mathbf{e}_k for the standard basis vectors, that is, \mathbf{e}_k , $k = 1, 2, \dots, n$ is the k th column of $n \times n$ identity matrix.

Denote \mathbf{A} as the transformations defined in Table 1. If μ is an eigenvalue of \mathbf{A}^2 , with multiplicity r , then $\lambda^+ = \sqrt{\mu}$ and $\lambda^- = -\sqrt{\mu}$ are possible eigenvalues of \mathbf{A} with multiplicities p and m , respectively, such that $p + m = r$ holds.

In the sequel, we will assume that n is large enough, i.e., $n \geq 2$ ($\text{DCT}_{(4)}$, $\text{DST}_{(4)}$, $\text{DCT}_{(8)}$, $\text{DST}_{(5)}$, and $\text{DST}_{(1)}$), $n \geq 4$ ($\text{DCT}_{(5)}$ and $\text{DST}_{(8)}$), and $n \geq 6$ for $\text{DCT}_{(1)}$. The remaining cases are straightforward and can be analyzed by hand.

5.1. DTT with two eigenvalues

Here we will analyze the DCT of type 4 and 8 and the DST of type 1, 4 and 5. In all considered cases, according to results presented in Sections 3.2, 3.4, 3.5, 3.6 and 3.7, square of the transformation matrix is proportional to the identity matrix \mathbf{I} ,

$$\mathbf{C}_{(4)}^2 = \mathbf{S}_{(4)}^2 = \frac{n}{2} \mathbf{I}, \quad \mathbf{C}_{(8)}^2 = \mathbf{S}_{(5)}^2 = \frac{2n+1}{4} \mathbf{I}, \quad \mathbf{S}_{(1)}^2 = \frac{n+1}{2} \mathbf{I},$$

resulting in eigenvalues of the corresponding matrices as in Table 2. Multiplicity of the positive and negative eigenvalue can be determined by calculating the trace of the transformation matrix. Denote by p the multiplicity of positive eigenvalue and by m the multiplicity of negative eigenvalue. We know that $p + m = n$ and that the trace of the transformation matrix is $(p - m)\lambda^+$, where λ^+ is the positive eigenvalue.

Using (11), (13), (14), (15), and (16) we can conclude that for even n , the traces of the transformation matrices are equal to zero, meaning that $p = m = n/2$. For odd n in each case we obtain that the trace of the transformation matrix is λ^+ , meaning that $p = m + 1$, that is, $p = (n + 1)/2$ and $m = (n - 1)/2$.

5.2. DCT type 5 case

Let us consider the following decomposition of \mathbb{R}^n into orthogonal subspaces \mathcal{V}_1 and \mathcal{V}_2

$$\mathbb{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where vectors in \mathcal{V}_1 are of the form

$$\mathbf{v} = [0, v_1, v_2, \dots, v_{n-1}]^T$$

with

$$\sum_{k=1}^{n-1} v_k = 0$$

and \mathcal{V}_2 is a two-dimensional space generated by

$$\mathbf{q}_1 = [1, 0, 0, \dots, 0]^T = \mathbf{e}_1$$

$$\mathbf{q}_2 = [0, 1, 1, \dots, 1]^T = \sum_{k=2}^n \mathbf{e}_k.$$

The square of the DCT type 5 transform matrix, $\mathbf{C}_{(5)}$, according to the results presented in Section 3.3, can be written as

$$\mathbf{C}_{(5)}^2 = \text{diag}_n \left(\frac{2n-1}{2}, \frac{2n-1}{4}, \dots, \frac{2n-1}{4} \right) + \frac{1}{2} \mathbf{1}_n, \quad (18)$$

where $\text{diag}_n(\cdot)$ is an $n \times n$ diagonal matrix and $\mathbf{1}_n$ is an $n \times n$ matrix with all ones. It is obvious that \mathcal{V}_1 is within null space of $\mathbf{1}_n$. By using (2) we have

$$\mathbf{C}_{(5)}^2 \mathbf{v} = \frac{2n-1}{4} \mathbf{v} \quad (\forall \mathbf{v} \in \mathcal{V}_1). \quad (19)$$

This proves $\frac{2n-1}{4} \in \text{Sp}(\mathbf{C}_{(5)}^2)$ and $\mathcal{V}_1 \subseteq \ker(\mathbf{C}_{(5)}^2 - \frac{2n-1}{4} \mathbf{I})$. Since $\mathbf{C}_{(5)} \pm \sqrt{\frac{2n-1}{4}} \mathbf{I}$ are non-singular, both $\sqrt{\frac{2n-1}{4}}$ and $-\sqrt{\frac{2n-1}{4}}$ are eigenvalues of $\mathbf{C}_{(5)}$. The dimension of \mathcal{V}_1 is $n - 2$, so the sum of the multiplicities is at least $n - 2$.

By focusing on \mathcal{V}_2 , we will find the other eigenvalues. We have that

$$\mathbf{C}_{(5)} \mathbf{q}_1 = \mathbf{q}_1 + \mathbf{q}_2 \quad (20)$$

$$\mathbf{C}_{(5)} \mathbf{q}_2 = (n-1) \mathbf{q}_1 - \frac{1}{2} \mathbf{q}_2. \quad (21)$$

The eigenvalues (and the corresponding eigenvectors) can be found by solving

$$\mathbf{C}_{(5)}(\mathbf{q}_1 + x \mathbf{q}_2) = \lambda(\mathbf{q}_1 + x \mathbf{q}_2)$$

for unknown λ (and x). Using (20) and (21) we get

$$(x(n-1) + 1) \mathbf{q}_1 + (1 - \frac{x}{2}) \mathbf{q}_2 = \lambda \mathbf{q}_1 + \lambda x \mathbf{q}_2,$$

resulting in a system of equations

$$\begin{aligned} x(n-1) + 1 &= \lambda \\ 1 - \frac{x}{2} &= \lambda x, \end{aligned}$$

and the eigenvalues

$$\lambda = \frac{1}{4} \pm \sqrt{n - \frac{7}{16}},$$

each with multiplicity one, as stated in Table 2. Moreover $(\frac{1}{4} \pm \sqrt{n - \frac{7}{16}})^2$ are distinct eigenvalues of $\mathbf{C}_{(5)}^2$, and recalling the $\mathcal{V}_1 \subseteq (\mathbf{C}_{(5)}^2 - \frac{2n-1}{4} \mathbf{I})$, we get

$$\text{Sp}(\mathbf{C}_{(5)}^2) = \left\{ \frac{2n-1}{4}, \frac{1}{4} + \sqrt{n - \frac{7}{16}}, \frac{1}{4} - \sqrt{n - \frac{7}{16}} \right\},$$

since $\dim \mathcal{V}_1 = n - 2$. The remaining eigenvalues of $\mathbf{C}_{(5)}$ are of the form $\sqrt{\frac{2n-1}{4}}$ or $-\sqrt{\frac{2n-1}{4}}$. Multiplicity of eigenvalues can be determined using the trace of the transformation matrix (12) and the fact that their multiplicities sum to $n - 2$. The sum of all eigenvalues is

$$\frac{1}{2} + (p - m) \sqrt{\frac{2n-1}{4}},$$

resulting in $p = m = n/2$ for an even n and $p = (n - 1)/2$, $m = (n - 3)/2$ for an odd n .

5.3. DST type 8 case

In analogy to the previous case, the subspace \mathcal{V}_1 is a set of vectors

$$\mathbf{v} = [v_0, v_1, v_2, \dots, v_{n-2}, 0]^T,$$

satisfying

$$\sum_{k=0}^{n-2} (-1)^k v_k = 0,$$

whereas \mathcal{V}_2 is a two-dimensional space generated by

$$\mathbf{q}_1 = [0, 0, \dots, 0, 1]^T = \mathbf{e}_n$$

$$\mathbf{q}_2 = [1, -1, 1, -1, \dots, (-1)^{n-2}, 0]^T = -\sum_{k=1}^{n-1} (-1)^k \mathbf{e}_k.$$

The square of $\mathbf{S}_{(8)}$ matrix, according to Section 3.8, can be written as

$$\mathbf{S}_{(8)}^2 = \text{diag}_n \left(\frac{2n-1}{4}, \frac{2n-1}{4}, \dots, \frac{2n-1}{2} \right) + \frac{1}{2} \mathbf{Q}. \quad (22)$$

The elements of matrix \mathbf{Q} are defined by $q_{kl} = (-1)^{kl}$. Again, the vector space \mathcal{V}_1 is within the null space of \mathbf{Q} resulting in

$$\mathbf{S}_{(8)}^2 \mathbf{v} = \frac{2n-1}{4} \mathbf{v} \quad (\forall \mathbf{v} \in \mathcal{V}_1).$$

Therefore, $\frac{2n-1}{4}$ is an eigenvalue of $\mathbf{S}_{(8)}^2$. One may conclude that both $\pm \sqrt{\frac{2n-1}{4}}$ are eigenvalues of $\mathbf{S}_{(8)}$. Moreover, the sum of multiplicities is at least $n - 2$.

The other eigenvalues can be found by solving

$$\mathbf{S}_{(8)}(\mathbf{q}_1 + x \mathbf{q}_2) = \lambda(\mathbf{q}_1 + x \mathbf{q}_2).$$

Using

$$\mathbf{S}_{(8)} \mathbf{q}_1 = (-1)^{n-1} \mathbf{q}_1 + \mathbf{q}_2$$

$$\mathbf{S}_{(8)} \mathbf{q}_2 = (n-1) \mathbf{q}_1 + \frac{(-1)^n}{2} \mathbf{q}_2,$$

we obtain the system of equations

$$(-1)^{n-1} + (n-1)x = \lambda$$

$$1 + \frac{(-1)^n}{2} x = \lambda x$$

with solutions

$$\lambda = -\frac{(-1)^n}{4} \pm \sqrt{n - \frac{7}{16}}.$$

Each eigenvalue has multiplicity one.

Now we can determine the multiplicities of all eigenvalues. Denoting by p the multiplicity of eigenvalue $\sqrt{(2n-1)/4}$ and with m the multiplicity of $-\sqrt{(2n-1)/4}$, where $p + m = n - 2$ we have that the sum of all eigenvalues is

$$(p - m)\sqrt{\frac{2n-1}{4}} - \frac{(-1)^n}{2}. \quad (23)$$

This sum is equal to the trace of $S_{(8)}$ matrix (17). For even n we have that the trace is $-\frac{1}{2}$, meaning that $p = m = \frac{n-2}{2}$. For an odd n we have that $p - m = 1$, resulting in $p = \frac{n-1}{2}$ and $m = \frac{n-3}{2}$, as given in Table 2.

5.4. DCT type 1 case

Similar to the previous types, to get the eigenvalues of $C_{(1)}$, we split the problem to some simpler components. In this case, we have to consider the odd and the even cases separately. Moreover \mathbb{R}^n is decomposed into three orthogonal subspaces,

$$\mathbb{R}^n = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3. \quad (24)$$

Let \mathcal{V}_1 be the $(n-4)$ -dimensional vector space containing vectors $\mathbf{v} = [0, v_1, v_2, \dots, v_{n-2}, 0]^T$ such that the sum of even indexed values is zero and the sum of odd indexed values is also zero

$$\sum_{k=1}^{(n-3)/2} v_{2k} = 0 \quad \text{and} \quad \sum_{k=1}^{(n-1)/2} v_{2k-1} = 0.$$

Note that, according to Section 3.1, we have

$$C_{(1)}^2 = \text{diag}(n-1, \frac{n-1}{2}, \frac{n-1}{2}, \dots, \frac{n-1}{2}, n-1) + \mathbf{P},$$

where \mathbf{P} is an $n \times n$ matrix with elements

$$p_{kl} = \frac{1 + (-1)^{k+l}}{2} = \begin{cases} 0 & \text{for odd } k+l \\ 1 & \text{for even } k+l. \end{cases}$$

One may directly check that \mathcal{V}_1 is contained in the null space of \mathbf{P} . Applying this point, we get that

$$C_{(1)}^2 \mathbf{v} = \frac{n-1}{2} \mathbf{v} \quad (\forall \mathbf{v} \in \mathcal{V}_1). \quad (25)$$

Thus, $(n-1)/2$ is an eigenvalue of $C_{(1)}^2$. Similar to the two previous type transforms, we conclude that $\pm\sqrt{(n-1)/2}$ are eigenvalues of $C_{(1)}$ corresponding to λ_3 and λ_4 in Table 3 for even and odd n .

Suppose that n is odd. Let us define the vector space \mathcal{V}_2 as a two-dimensional space generated by

$$\mathbf{q}_{21} = [1, 0, 0, \dots, 0, -1]^T = \mathbf{e}_1 - \mathbf{e}_n \quad (26)$$

$$\mathbf{q}_{22} = [0, 1, 0, 1, 0, \dots, 0, 1, 0]^T = \sum_{k=1}^{(n-1)/2} \mathbf{e}_{2k}, \quad (27)$$

and \mathcal{V}_3 as a two-dimensional space generated by

$$\mathbf{q}_{31} = [1, 0, 0, \dots, 0, 1]^T = \mathbf{e}_1 + \mathbf{e}_n \quad (28)$$

$$\mathbf{q}_{32} = [0, 0, 1, 0, 1, \dots, 1, 0, 0]^T = \sum_{k=1}^{(n-3)/2} \mathbf{e}_{2k+1}. \quad (29)$$

Consider subspace \mathcal{V}_2 . We have that Not obvious

$$C_{(1)} \mathbf{q}_{21} = 2\mathbf{q}_{22} \quad (30)$$

$$C_{(1)} \mathbf{q}_{22} = \frac{n-1}{2} \mathbf{q}_{21}. \quad (31)$$

Let us now find eigenvalues and eigenvectors within this subspace. We should find x such that

$$C_{(1)}(\mathbf{q}_{21} + x\mathbf{q}_{22}) = \lambda(\mathbf{q}_{21} + x\mathbf{q}_{22})$$

for some λ . Then $\mathbf{q}_{21} + x\mathbf{q}_{22}$ is an eigenvector in \mathcal{V}_2 and λ is the corresponding eigenvalue. Using (30) and (31) we get

$$2\mathbf{q}_{22} + x \frac{n-1}{2} \mathbf{q}_{21} = \lambda \mathbf{q}_{21} + \lambda x \mathbf{q}_{22},$$

resulting in a system of equations with unknown x and λ

$$\begin{aligned} 2 &= \lambda x \\ x \frac{n-1}{2} &= \lambda. \end{aligned}$$

Solutions to this system are the eigenvalues $\lambda = \pm\sqrt{n-1}$ corresponding to λ_1 and λ_5 in Table 3 for an odd n . Each eigenvalue has multiplicity one.

Consider now \mathcal{V}_3 space. Similar to the previous case we get Not obvious

$$C_{(1)} \mathbf{q}_{31} = 2\mathbf{q}_{31} + 2\mathbf{q}_{32}$$

$$C_{(1)} \mathbf{q}_{32} = \frac{n-3}{2} \mathbf{q}_{31} - \mathbf{q}_{31}.$$

Next, we search for eigenvectors and eigenvalues from

$$C_{(1)}(\mathbf{q}_{31} + x\mathbf{q}_{32}) = \lambda(\mathbf{q}_{31} + x\mathbf{q}_{32}),$$

leading to the system of equations

$$\begin{aligned} 2 + \frac{n-3}{2} &= \lambda \\ 2 - x &= \lambda x, \end{aligned}$$

with the solutions

$$\lambda = \frac{1}{2} \pm \sqrt{n - \frac{3}{4}},$$

corresponding to λ_2 and λ_6 in Table 3. Both of the obtained eigenvalues are with multiplicity one.

Note that in the considered cases we have an analytical form for the corresponding eigenvectors.

Now we can determine multiplicities of all eigenvalues. Denoting by p the multiplicity of eigenvalue $\sqrt{(n-1)/2}$ and by m the multiplicity of $-\sqrt{(n-1)/2}$, where $p+m = n-4$, we have that the sum of all eigenvalues is

$$(p - m)\sqrt{\frac{n-1}{2}} + 1. \quad (32)$$

This sum is equal to the trace of $C_{(1)}$ matrix (10). We can conclude that $p - m = 1$, resulting in $p = \frac{n-3}{2}$ and $m = \frac{n-5}{2}$, as given in Table 3 for odd n case.

Now we will consider the case of an even n . In decomposition (24) \mathcal{V}_1 remains the same, while \mathcal{V}_2 is now spanned by vectors

$$\mathbf{q}_{21} = [1, 0, 0, \dots, 0, \sqrt{2} - 1] = \mathbf{e}_1 + (\sqrt{2} - 1)\mathbf{e}_n \quad (33)$$

$$\begin{aligned} \mathbf{q}_{22} &= [0, 1, \sqrt{2} + 1, \dots, \sqrt{2} + 1, 0] \\ &= \sum_{k=1}^{n-2/2} \mathbf{e}_{2k} + (\sqrt{2} + 1) \sum_{k=1}^{n-2/2} \mathbf{e}_{2k+1} \end{aligned} \quad (34)$$

and \mathcal{V}_3 is spanned by

$$\mathbf{q}_{31} = [1, 0, 0, \dots, 0, -(\sqrt{2} + 1)] = \mathbf{e}_1 - (\sqrt{2} + 1)\mathbf{e}_n \quad (35)$$

$$\begin{aligned} \mathbf{q}_{32} &= [0, 1, 1 - \sqrt{2}, \dots, 1 - \sqrt{2}, 0] \\ &= \sum_{k=1}^{n-2/2} \mathbf{e}_{2k} + (1 - \sqrt{2}) \sum_{k=1}^{(n-2)/2} \mathbf{e}_{2k+1}. \end{aligned} \quad (36)$$

It is easy to check that \mathcal{V}_2 and \mathcal{V}_3 are invariant with respect to $C_{(1)}$, that is, Not obvious

$$C_{(1)} \mathbf{q}_{21} = \sqrt{2}\mathbf{q}_{21} + (2 - \sqrt{2})\mathbf{q}_{22} \quad (37)$$

$$C_{(1)} \mathbf{q}_{22} = (n-2)(1 + \frac{\sqrt{2}}{2})\mathbf{q}_{21} - \frac{\sqrt{2}}{2} \mathbf{q}_{22} \quad (38)$$

$$C_{(1)} \mathbf{q}_{31} = -\sqrt{2}\mathbf{q}_{31} + (2 + \sqrt{2})\mathbf{q}_{32} \quad (39)$$

$$\mathbf{C}_{(1)}\mathbf{q}_{32} = -(n-2)\frac{\sqrt{2}}{2}\mathbf{q}_{31} + \frac{\sqrt{2}}{2}\mathbf{q}_{32}. \quad (40)$$

The eigenvalues (with corresponding eigenvectors) can be found by solving the system

$$\mathbf{C}_{(1)}(\mathbf{q}_{21} + x\mathbf{q}_{22}) = \lambda(\mathbf{q}_{21} + x\mathbf{q}_{22}) \quad (41)$$

$$\mathbf{C}_{(1)}(\mathbf{q}_{31} + x\mathbf{q}_{32}) = \lambda(\mathbf{q}_{31} + x\mathbf{q}_{32}), \quad (42)$$

for unknown λ (and x). Form (41), using (37) and (38), we obtain the system of equations

$$\begin{aligned} \sqrt{2} + x(n-2)(1 + \frac{\sqrt{2}}{2}) &= \lambda \\ 2 - \sqrt{2} - x\frac{\sqrt{2}}{2} &= \lambda x \end{aligned}$$

with solutions

$$\lambda = \frac{\sqrt{2}}{4} \pm \sqrt{n - \frac{7}{8}},$$

corresponding to λ_2 and λ_6 in Table 3 for even n . In a similar way, by solving (42) we can obtain λ_1 and λ_5 .

The sum of all eigenvalues is

$$(p-m)\sqrt{\frac{n-1}{2}}, \quad (43)$$

and the trace of $\mathbf{C}_{(1)}$ is zero, (10), resulting in $p = m = (n-4)/2$, as stated in Table 3 for an even n .

6. Conclusion

An analytic proof for eigenvalues, and corresponding multiplicities is provided for eight symmetric non-normalized DTT. The trace and the square of transformation matrix is derived in all analyzed cases. Our further research will include derivation of eigenvector basis for the analyzed DTT. The proposed approach, based on the decomposition of the eigenspace into orthogonal subspaces invariant under considered DTT, provides eigenvalues and some eigenvectors, for the case when the eigenvalue multiplicity is 1. Since the DTT are fundamental mathematical tools for signal processing and related applications, we believe that the presented theory is particularly relevant to this field, since it sheds a new light on the understanding of commonly used transforms.

Finding an appropriate approximation of the eigenvalues of non-symmetric types of DTTs is a complicated and challenging problem. The key point that drove the discussion forward effectively was the fact that the minimal polynomial of all symmetric DTTs is of degree at most 6 for all sizes. However, this key point no longer holds for non-symmetric types, requiring an alternative approach to determine the eigenvalues within this context.

In the literature, efforts have been dedicated to computing the eigenvectors of symmetric DTTs and devising a method to extract the eigenvalues and their corresponding multiplicities. Nevertheless, this approach is limited to specific types. Using sampling theory, the eigenvectors are derived from the offset discrete Fourier transform of the corresponding counterpart. Notably, it has been observed that the eigenvalues of the offset discrete Fourier transform [16], associated with non-symmetric types of DTTs, are all distinct. This observation suggests that the eigenvalues of non-symmetric types of DTTs are also distinct, a hypothesis supported by empirical evidence.

The eigenvalues of a discrete trigonometric transforms have several applications in the context of signal and image compression. In [10] the role of eigenvalues in determining the significance of frequency components during compression is discussed.

CRedit authorship contribution statement

Ali Bagheri Bardi: Conceptualization, Formal analysis, Methodology, Writing – original draft. **Miloš Daković:** Conceptualization, Formal analysis, Methodology, Software, Validation, Writing – original draft. **Taher Yazdanpanah:** Formal analysis, Methodology, Writing – review & editing. **Ljubiša Stanković:** Formal analysis, Methodology, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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