Matched Filtering on Directed Graphs

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***Abstract*—The matched filter is a crucial concept in both signal analysis and convolutional neural networks (CNNs). Previous work has addressed graph matched filtering principles for undirected graphs. This paper expands upon the existing literature, by exploring matched filtering principles for signals on directed graphs. In such cases, the adjacency matrix is asymmetric, and commonly results in nonorthogonal eigenvectors. The presented concept is supported by a detailed analysis and numerical examples.**

***Index Terms*—matched filter, CNN, graph signal processing, asymmetric adjacency matrices, directed graphs**

I. Introduction

Signal processing on graphs has become an increasingly prominent research area, since it can be viewed as a generalization of the traditional theory of signal processing to irregular signal domains [1]–[14]. Despite the previously acknowledged significance of graphs for the development of machine learning and artificial intelligence, the theory behind graph signal processing is still being established. For example, even some key concepts, such as matched filtering on graphs or the interpretation of convolutional neural networks (CNN), remains rather incomplete [15], [16]. In our recent work [17], [18], it was shown that matched filtering underpins the operation of CNNs; therefore understanding their concept and interpretation is crucial for effectively utilizing CNNs. We also showed that the fundamental operations of graph CNNs (GCNNs), such as the convolution-activation-pooling chain, can be easily understood through the framework of matched filtering [17], thus providing a basis to improve the effectiveness of all future work in artificial intelligence and machine learning.

Recent works in [17], [18] also explored the concept of matched filtering as the platform for the understanding of convolutional layers in GCNNs. In [19], the development of the convolutional layers in deep neural networks was interpreted using signal processing tools, with matched filters among others. The application of CNNs classifiers for real-time ECG using matched filters was presented in [20]. In [21], [22], the matched filter used in neural networks was applied to gravitational waves, while in [23], the multiplex form of graph matched filtering was introduced. However, these studies have only been conducted for the case of undirected graphs. Directed graphs have recently gained significant attention in theory and applications [24], [25] and the generalization of the matched filtering to these graphs is of crucial importance for the understanding of their operation but is still missing. In the sense of representation, an undirected graph can be seen as a special case of directed graphs.

To fill this void in the literature, we set out to extend the theory of matched filtering to general graphs, with both symmetric and asymmetric shift matrices. In directed graphs, as a general rule, the corresponding eigenvectors of the graph topology matrix are not orthogonal, which makes the development of the matched filtering theory quite complex and not straightforward. The aim of this paper is therefore to expand the matched filtering theory to all diagonalizable graph connectivity matrices.

The paper is organized as follows. The Introduction is presented in Section 1. The theoretical background on signal processing on graphs is discussed in Section 2. In Section 3, matched filters on graphs are presented. Section 4 contains numerical examples, while Section 5 concludes the paper.

II. Theoretical background

Recent advances in graph signal processing and machine learning have reflected a growing interest in graph-based representation of irregular signal domains [1]–[4], [10]. A graph consists of a set of vertices, representing a set of points where the signal is sensed, and a set of edges, which reflect the relations between the vertices. Graphs may be undirected, whereby the relation between two vertices is mutual, or directed, when the connectivity between two vertices is, in general, one-sided. The edges may be: (i) unweighted, which is modeled by the adjacency matrix, $\mathbf{A}$, whose elements are 1 (indicating that two vertices are connected), or 0 (two vertices are not connected); or (ii) weighted, that is, modeled by the weight matrix $\mathbf{W}$ (whose values usually depend on the weights of the vertex connectivity). Without loss of generality, we assume that the graphs are represented by their adjacency matrix $\mathbf{A}$, which is diagonalizable. Note that any other shift operator, such as the weight matrix, graph Laplacian [26], or random walk can be also used.

The eigendecomposition of the real-valued, diagonalizable, adjacency matrix, $\mathbf{A}$, of a graph is given by

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}, \tag{1}$$

where $\mathbf{U}$ is a matrix consisting of the eigenvectors $\mathbf{u}_k$, $k = 1, 2, \dots, N$, with the eigenvalues, λ_k, comprised in the diagonal matrix $\mathbf{\Lambda}$. The adjacency matrix of directed graphs is, in general, not symmetric, thus yielding the complex-valued (typically non-orthogonal) eigenvectors and eigenvalues.

This work is partially supported by the Montenegrin Ministry of Education, Science and Innovations, project grant: DPG "Data Processing on Graphs", and by the Montenegrin Academy of Sciences and Arts.

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This article has been accepted for publication in IEEE Transactions on Systems, Man, and Cybernetics: Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TSMC.2024.3407948

The graph Fourier transform (GFT) of a signal, $x(n)$, on vertices $n = 1, 2, \dots, N$, is defined as

$$X(k) = \sum_{n=1}^{N} x(n) v_k(n), \tag{2}$$

where $v_k(n)$ represents the value of the kth column of matrix $\mathbf{U}^{-1}$, corresponding to a vertex n. The GFT in (2) can also be written in a matrix form, as $\mathbf{X} = \mathbf{U}^{-1}\mathbf{x}$.

The inverse graph Fourier transform (IGFT) is defined by

$$x(n) = \sum_{k=1}^{N} X(k) u_k(n), \tag{3}$$

or, in a matrix form, as $\mathbf{x} = \mathbf{U}\mathbf{X}$, where $u_k(n)$ is the value of the kth eigenvector (column of matrix $\mathbf{U}$), corresponding to a vertex n.

The undirected graph can be considered as *a special case* in terms of the adjacency matrix $\mathbf{A}$. Its adjacency matrix is always symmetric. For graphs with symmetric matrices $\mathbf{A}$, the eigenvectors are real-valued and $v_k(n) = u_k(n)$, that is $\mathbf{U}^{-1} = \mathbf{U}^T$, and $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix. It is important to note that, for directed graphs, $\mathbf{A} \neq \mathbf{A}^T$.

For a very specific case of asymmetric normal matrices $v_k(n) = u_k^*(n)$ and $\mathbf{U}^{-1} = \mathbf{U}^H$ holds, where $(\cdot)^H$ denotes the Hermite transpose operator.

III. White noise on a graph

Consider a random, real-valued, white noise input signal, $\varepsilon(n)$, on a graph, with the correlation function

$$r_\varepsilon(n, m) = \mathrm{E}\{\varepsilon(n)\varepsilon(m)\} = \sigma_\varepsilon^2 \delta(n - m) \tag{4}$$

and $\mathrm{E}\{\varepsilon(n)\} = 0$. Here, $n, m \in \{1, 2, \dots, N\}$ denote two vertex indices. The fact that $r_\varepsilon(n, m) = \sigma_\varepsilon^2$ for $n = m$ and $r_\varepsilon(n, m) = 0$ for $n \neq m$ signifies that white noise samples exhibit no correlation. The GFT of this signal is given by (2)

$$\mathcal{E}(k) = \sum_{n=1}^{N} \varepsilon(n) v_k(n). \tag{5}$$

The autocorrelation function of the GFT of white noise is

$$\begin{aligned} r_{\mathcal{E}}(k, l) &= \mathrm{E}\{\mathcal{E}(k)\mathcal{E}^*(l)\} \\ = \sum_{n=1}^{N}\sum_{m=1}^{N} \mathrm{E}\{\varepsilon(n)\varepsilon^*(m)\} v_k(n) v_l^*(m) &= \sigma_\varepsilon^2 \sum_{n=1}^{N} v_k(n) v_l^*(n) \\ &= \sigma_\varepsilon^2 r_v(k, l), \end{aligned} \tag{6}$$

while the autocorrelation matrix is $\mathbf{R}_\varepsilon = \sigma_\varepsilon^2 (\mathbf{U}^{-1})^H \mathbf{U}^{-1}$. For the orthogonal transformation matrices $\mathbf{U}^H\mathbf{U} = \mathbf{I}$, when $r_v(k, l) = \delta(k - l)$, the GFT values also represent white random noise. However, that is not true for the a case of transformation matrices in directed graphs, where the noise in the spectral domain is correlated.

IV. System on a graph and graph convolution

A system on a graph is defined as

$$\mathbf{y} = h_0\mathbf{x} + h_1\mathbf{A}\mathbf{x} + \cdots + h_{M-1}\mathbf{A}^{M-1}\mathbf{x}, \tag{7}$$

with $\mathbf{x}$ as the input graph signal, and $h_0, h_1, \dots, h_{M-1}$ as system coefficients. The spectral domain representation is obtained using (1) and by multiplying the right side of the vertex domain relation (7) by $\mathbf{U}^{-1}$. This gives

$$\mathbf{Y} = H(\boldsymbol{\Lambda})\mathbf{X} = (h_0 + h_1\boldsymbol{\Lambda} + \cdots + h_{M-1}\boldsymbol{\Lambda}^{M-1})\mathbf{X}, \tag{8}$$

where $\mathbf{X} = \mathbf{U}^{-1}\mathbf{x}$ and $\mathbf{Y} = \mathbf{U}^{-1}\mathbf{y}$ are the GFTs of $\mathbf{x}$ and $\mathbf{y}$, respectively, and $H(\boldsymbol{\Lambda})$ is a diagonal matrix representing the transfer function of the system.

In the spectral domain, graph convolution is defined by the element-wise product of the system, i.e. $Y(k) = H(\lambda_k)X(k)$. In the vertex domain, the graph convolution is calculated as the inverse GFT of the output, that is, $y(n) = x(n) * h(n) = \mathrm{IGFT}\{X(k)H(\lambda_k)\}$ [2].

Note that, in order to emphasize the distinction between the GFT of a signal, $X(k)$ and $Y(k)$, and the transfer function in the spectral domain, we denote the k-th coefficient of the transfer function as $H(\lambda_k)$ [3].

V. Graph matched filter (GMF)

Consider an input graph signal, $x(n)$, given by

$$x(n) = s(n) + \varepsilon(n),$$

where $s(n)$ is the desired signal and $\varepsilon(n)$ is white noise. The system through which the signal is processed can be described by the transfer function $G(\lambda_k)$, with

$$g(n) = \mathrm{IGFT}\{G(\lambda_k)\}.$$

The output, $y(n) = y_s(n) + y_\varepsilon(n)$, is then calculated as

$$\begin{aligned} y(n) &= x(n) * g(n) = (s(n) + \varepsilon(n)) * g(n) \\ &= \mathrm{IGFT}\{S(k)G(\lambda_k)\} + \mathrm{IGFT}\{\mathcal{E}(k)G(\lambda_k)\} \\ &= \sum_{k=1}^{N} S(k)G(\lambda_k)u_k(n) + \sum_{k=1}^{N} \mathcal{E}(k)G(\lambda_k)u_k(n) \\ &= y_s(n) + y_\varepsilon(n). \end{aligned}$$

The goal of the matched filter is to find the transfer function that maximizes the power of the output graph signal for an input graph signal $x(n)$, at a vertex $n = n_0$. The ratio of the signal and noise power at a vertex n_0 is given by

$$\frac{|y_s(n_0)|^2}{\mathrm{E}\Big\{|y_\varepsilon(n_0)|^2\Big\}} = \frac{\Big|\sum_{k=1}^{N} S(k)G(\lambda_k)u_k(n_0)\Big|^2}{\mathrm{E}\Big\{\Big|\sum_{k=1}^{N} \mathcal{E}(k)G(\lambda_k)u_k(n_0)\Big|^2\Big\}}. \tag{9}$$

For the component $y_s(n)$ of the output of the system on a graph which is related only to the desired signal, $s(n)$, at a vertex n_0 we may write

$$\begin{aligned} |y_s(n_0)|^2 &= \Big|\sum_{k=1}^{N} S(k)G(\lambda_k)u_k(n_0)\Big|^2 \\ &\leq \sum_{k=1}^{N} |S(k)|^2 \sum_{k=1}^{N} |G(\lambda_k)u_k(n_0)|^2. \end{aligned} \tag{10}$$

The Schwartz inequality states that the maximum is achieved when the equality holds, which will yield the transfer function of the matched filter satisfying

$$G(\lambda_k)u_k(n_0) = S^*(k), \tag{11}$$

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up to a possible scaling constant, with $S^*(k)$ as the complex conjugate of $S(k)$. The maximum value of the output is then

$$y(n_0) = \sum_{k=1}^{N} |S(k)|^2 = \mathbf{S}^H\mathbf{S} = \mathbf{s}^H(\mathbf{U}^{-1})^H\mathbf{U}^{-1}\mathbf{s} = E_s^f, \quad (12)$$

where E_s^f is the input signal energy in the spectral domain. Unlike for graphs with symmetric adjacency matrices, for the asymmetric ones, with nonorthogonal $\mathbf{U}^{-1}$, the Parseval theorem in general does not hold. This energy is not the same as the vertex domain energy, that is

$$E_s = \sum_{n=1}^{N} |s(n)|^2 = \mathbf{s}^H\mathbf{s} = \mathbf{S}^H\mathbf{U}^H\mathbf{U}\mathbf{S}. \quad (13)$$

The output signal-to-noise power, for $G(\lambda_k)u_k(n_0) = S^*(k)$ and $\mathrm{E}\{\mathcal{E}(k)\mathcal{E}^*(l)\} = \sigma_\varepsilon^2 r_v(k,l) = \sigma_x^2 \sum_{n=1}^{N} v_k(n)v_l^*(n)$, is given by

$$\frac{|y_s(n_0)|^2}{\mathrm{E}\Big\{|y_\varepsilon(n_0)|^2\Big\}} = \frac{\sum_{k=1}^{N}|S(k)|^2\sum_{k=1}^{N}|S(k)|^2}{\sigma_\varepsilon^2\sum_{k=1}^{N}\sum_{l=1}^{N} r_v(k,l)S(k)S^*(l)} = \frac{(E_s^f)^2}{E_s^{fv}\sigma_\varepsilon^2},$$

where $E_s^{fv} = \mathbf{S}^H(\mathbf{U}^{-1})^H\mathbf{U}^{-1}\mathbf{S}$ would be the energy in the spectral domain if the GFT values $S(k)$ were assumed as the graph signal, in the sense of (12). For orthogonal matrices, when Parseval's theorem holds, that is, when $E_s^f = E_s = E_s^{fv}$, the output signal-to-noise power reduces to the well-known value of E_s/σ_ε^2.

A. Graph matched filter for a diffusion signal

For a clearer interpretation of the solution, we shall assume that the graph signal is obtained from an $(M-1)$-step diffusion. The initial signal is a delta pulse at a vertex n_0 [18], which can be defined as $s_0(n) = \delta(n-n_0)$. A vector form of this signal is denoted by $\mathbf{s}_0$. Its GFT is

$$S_0(k) = \sum_{n=1}^{N} s_0(n)v_k(n) = v_k(n_0).$$

When the graph shift operator is applied $(M-1)$ times, the graph signal, $\mathbf{s}$, is obtained as

$$\mathbf{s} = a_0\mathbf{s}_0 + a_1\mathbf{A}\mathbf{s}_0 + \cdots + a_{M-1}\mathbf{A}^{M-1}\mathbf{s}_0, \quad (14)$$

where a_p, $p = 0, 1, \ldots, M-1$, are the diffusion system coefficients. Note that we have used the adjacency matrix $\mathbf{A}$ for the notation of the shift operator, but any other shift operator can be used. The GFT of signal $\mathbf{s}$ can be written as

$$\mathbf{S} = \Big(a_0 + a_1\mathbf{\Lambda} + \cdots + a_{M-1}\mathbf{\Lambda}^{M-1}\Big)\mathbf{S}_0. \quad (15)$$

Considering the element-wise form of (15),

$$S(k) = \Big(a_0 + a_1\lambda_k + \cdots + a_{M-1}\lambda_k^{M-1}\Big)v_k(n_0), \quad (16)$$

the spectral representation of matched filter (11) follows from

$$G(\lambda_k)u_k(n_0) = \Big(a_0 + a_1\lambda_k^* + \cdots + a_{M-1}(\lambda_k^*)^{M-1}\Big)v_k^*(n_0). \quad (17)$$

Note that this form is vertex-dependent (depends on n_0) and will be elaborated upon within simulation examples.

In general, the output to the signal $x(n)$, with the GFT $X(k)$, filtered by the matched filter $G(\lambda_k)$, is

$$y(n) = \sum_{k=1}^{N} G(\lambda_k)X(k)u_k(n) = \sum_{k=1}^{N} G(\lambda_k)u_k(n)\; X(k).$$

For $G(\lambda_k)u_k(n)$ satisfying (17) we obtain

$$y(n) = \sum_{k=1}^{N}\Big(a_0 + a_1\lambda_k^* + \cdots + a_{M-1}(\lambda_k^*)^{M-1}\Big)v_k^*(n)X(k)$$
$$= \sum_{k=1}^{N}\Big(a_0 + a_1\lambda_k + \cdots + a_{M-1}(\lambda_k)^{M-1}\Big)v_k(n)X^*(k),$$

since the graph signal $y(n)$ is real-valued. A vector/matrix form of the matched filtering relation is given by

$$\mathbf{y} = (\mathbf{U}^{-1})^T\Big(a_0\mathbf{I} + a_1\mathbf{\Lambda} + \cdots + a_{M-1}\mathbf{\Lambda}^{M-1}\Big)\mathbf{X}^*$$
$$= \Big(a_0\mathbf{I} + a_1\mathbf{A}^T + \cdots + a_{M-1}(\mathbf{A}^T)^{M-1}\Big)(\mathbf{U}^{-1})^T(\mathbf{U}^{-1}\mathbf{x})^*$$
$$= \Big(a_0\mathbf{I} + a_1\mathbf{A}^T + \cdots + a_{M-1}(\mathbf{A}^T)^{M-1}\Big)(\mathbf{U}^*\mathbf{U}^T)^{-1}\mathbf{x}. \quad (18)$$

Modified forms and special cases: If we desire that the maximum value of the output signal is equal to the vertex domain energy, a slight modification of the matched filter is needed. Its form is

$$\mathbf{y} = \Big(a_0\mathbf{I} + a_1\mathbf{A}^T \cdots + a_{M-1}(\mathbf{A}^T)^{M-1}\Big)\mathbf{x}. \quad (19)$$

In this case, we have also to ensure that the shifts do not increase signal energy, meaning that it is better to use normalized adjacency matrices $\mathbf{A} \leftarrow \mathbf{A}/\lambda_{\max}$, where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{A}$.

Special cases for specific adjacency matrices are:

1) Asymmetric but normal, $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$, when $\mathbf{U}\mathbf{U}^H = \mathbf{I}$ and $\mathbf{U}^*\mathbf{U}^T = (\mathbf{U}\mathbf{U}^H)^T = \mathbf{I}$, then
$$\mathbf{y} = \Big(a_0\mathbf{I} + a_1\mathbf{A}^T + \cdots + a_{M-1}(\mathbf{A}^T)^{M-1}\Big)\mathbf{x}. \quad (20)$$
2) Symmetric, $\mathbf{A}^T = \mathbf{A}$. This matrix corresponds to undirected graphs. A symmetric matix is always diagonalizable, that is $\mathbf{U}\mathbf{U}^H = \mathbf{I}$. In this case, $Y(k) = G(\lambda_k)X(k)$ and (18) reduces to
$$\mathbf{y} = \Big(a_0\mathbf{I} + a_1\mathbf{A} + \cdots + a_{M-1}\mathbf{A}^{M-1}\Big)\mathbf{x}. \quad (21)$$

B. Discussion on applications

Matched filters can be used to find subgraphs within larger graphs that closely resemble a given pattern, which is crucial in areas such as bioinformatics (e.g. finding specific patterns or motifs in biological networks) and social network analysis, to name but a few. By convolving a matched filter with the target graph, regions of high similarity can be identified [28]. In directed graphs representing flow networks (e.g., traffic flow, information flow), matched filter can be used to detect patterns of flow or identify critical pathways [29]. In causal networks represented as directed graphs, matched filters can

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help identify causal relationships or patterns of influence between variables [30]. Directed graphs with temporal dynamics can also benefit from matched filters, as matched filters can capture temporal patterns of interactions or information flow over time.

In GCNNs, the whole convolution-activation-pooling chain can be explained from the perspective of matched filters. This viewpoint offers deep insights into the functionality of GCNNs, potentially addressing their interpretability and explainability challenges [17], [18]. Moreover, the matched filter coefficients can be applied for the initialization of convolution layer filters in GCNNs. By doing so, the filters are initialized with coefficients that are specifically tailored to capture relevant features or patterns in the graph data.

VI. Example

Consider a directed graph with an (asymmetric) adjacency matrix, shown in Fig. 1. In order to prevent the energy increase by shifts, we used $\mathbf{A}/\lambda_{\max} \to \mathbf{A}$, [27].

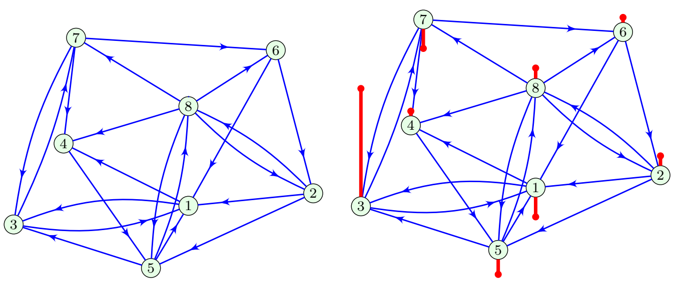

Fig. 1. An asymmetric graph with unit weights (left) and a graph signal obtained by a diffusion of $s_0(n) = \delta(n-3)$ in three steps (right).

We consider a graph signal created through a diffusion by graph shifting a delta pulse, $s_0(n) = \delta(n-3)$, as

$$\mathbf{s} = 2\mathbf{s}_0 - \mathbf{A}\mathbf{s}_0 + 0.5\mathbf{A}^2\mathbf{s}_0 + 0.1\mathbf{A}^3\mathbf{s}_0. \tag{22}$$

which is shown in Fig.1 (right).

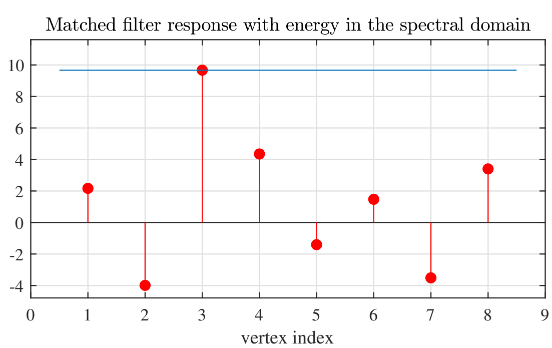

Fig. 2. Matched filter output for the signal from Fig. 1 (right). The signal energy in the spectral domain is indicated by the blue horizontal line.

The matched filter response to this signal is calculated using (18) and shown in Fig. 2. Its maximum is equal to the energy in the spectral domain, indicated by the blue horizontal line.

When the matched filter is modified as in (19), its maximum response is equal to the energy in the vertex domain, as shown in Fig. 3.

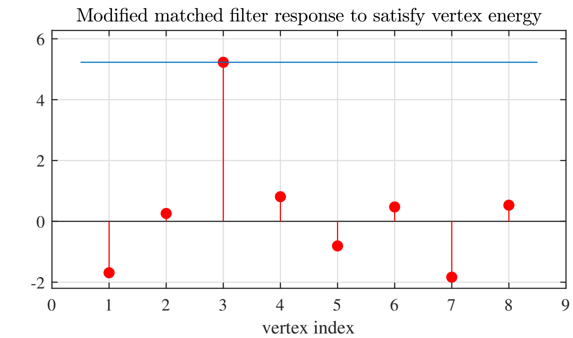

Fig. 3. Modified matched filter output for the signal from Fig. 1 (right). The signal energy in the vertex domain is indicated by the blue horizontal line.

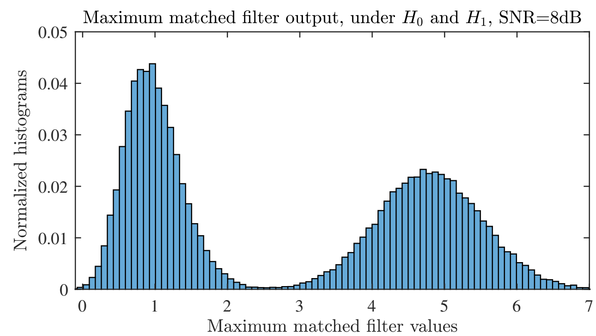

Fig. 4. Histogram of the matched filter output when the desired signal exists (left) and when it does not exist (right) in the input signal.

Statistical analysis. All the results were checked statistically. We considered the cases: $H_0 : x(n) = \varepsilon(n)$ and $H_1 : x(n) = s(n) + \varepsilon(n)$. The maximum value of the matched filter output was found over 100,000 realizations. Fig. 4, shows the limit case, when the histograms under H_0 and H_1 touch each other, with the SNR of 8 dB. Note that the number of samples is very low, $N = 8$. For larger SNR, histograms are well separated, while for lower SNR values the degree of overlapping increases.

Approximative model. Next, we assume that the signal was obtained by a higher-order diffusion process, which does not correspond to the assumed, for example second order, matched filter. Here, a graph signal was created through a diffusion by graph shifting a delta pulse, $s_0(n) = \delta(n-3)$, as

$$\mathbf{s} = \Big(\mathbf{I} - 0.8\mathbf{A} + 0.6\mathbf{A}^2 + 0.2\mathbf{A}^3 + 0.1\mathbf{A}^4 + 0.1\mathbf{A}^5\Big)\mathbf{s}_0 + 0.15\boldsymbol{\epsilon}$$

where ϵ contains samples of a unit variance zero-mean Gaussian noise, $\varepsilon(n)$. This system is well above the second order.

In this case, we found the graph Fourier transform of the signal that we are looking for, as $\mathbf{S} = \mathbf{U}\mathbf{s}$, and then approximated, in the LS sense, this transform by a third order system as in (16), $\Big(\hat{a}_0 + \hat{a}_1\lambda_k + \hat{a}_2\lambda_k^2\Big)v_k(n_0) = S(k)$,

$$[\hat{a}_0 \;\; \hat{a}_1 \;\; \hat{a}_2]^T = (\mathbf{V}^H\mathbf{V})^{-1}\mathbf{V}^H\mathbf{S}, \tag{23}$$

where the $N \times 3$ matrix $\mathbf{V}$ has elements $v_k(n_0)\lambda_k^m$, $k = 1, 2, \ldots, N$, $m = 0, 1, 2$.

Initial vertex position unknown. If the initial vertex for the diffusion is unknown, then the same procedure should

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be applied for each $n_0 = 1, 2, \dots, N$. Signals $\mathbf{s}_0$ are formed with $s(n) = 1$, for each $n = n_0$, $n_0 = 1, 2, \dots, N$. The corresponding sets of $\hat{a}_i$, $i = 0, 1, 2$, were calculated for these values of n_0 in $v_k(n_0)\lambda_k^m$. The set of $\hat{a}_i$, when the best fit to the original signal was achieved, that is, when $\arg\{\min_{n_0} ||\mathbf{s} - (\hat{a}_0\mathbf{I} + \hat{a}_1\mathbf{A} + \hat{a}_2\mathbf{A}^2)\mathbf{s}_0)||\} \to n_0$, was used in the matched filter, for example using (19), as

$$\mathbf{y} = \Big(\hat{a}_0\mathbf{I} + \hat{a}_1\mathbf{A}^T + \hat{a}_2(\mathbf{A}^T)^2)\mathbf{s}.$$

The response of this approximate matched filter, to a signal obtained by a higher-order diffusion process than the matched filter order, with an unknown initial vertex v_0, is shown in Fig. 5.

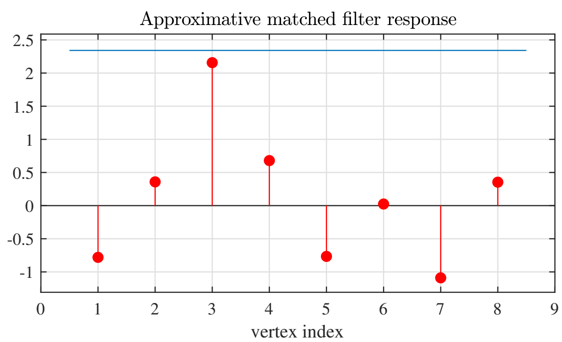

Fig. 5. Modified matched filter output for the signal which is not obtained through a process that exactly corresponds to the matched filter-based model (but it is approximated by a diffusion system of an assumed order). The signal energy in the vertex domain is indicated by the blue horizontal line.

Observe that the maximum response of the matched filter was not equal to the vertex energy of the signal, due to the lower-order approximation of the diffusion process, but it was quite close, with similar statistical behavior.

VII. Conclusions

We have extended the principle of matched filtering for signals on directed graphs. The adjacency matrix in this case is asymmetric, and with nonorthogonal eigenvector matrices. The analysis has generalized the matched filtering principle to all diagonizable adjacency matrices. The theory is supported by numerical examples.

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This article has been accepted for publication in IEEE Transactions on Systems, Man, and Cybernetics: Systems. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TSMC.2024.3407948

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