A Robust Eigenbasis Generation System for the Discrete Fourier Transform

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Abstract

A method has been developed for generating a comprehensible real eigenbasis for discrete Fourier transforms through a direct conversion of the eigenbases of discrete sine and cosine transforms. An advantageous outcome of employing these types of eigenbases is the emergence of a discrete version of Hermite functions.

Keywords: Discrete Fourier transform, Eigendecomposition, Real eigenvectors

1. Introduction

The discrete (continuous) Fourier transform stands out as a pivotal element in both electrical engineering and mathematics. It occupies a central role in signal processing and constitutes a fundamental aspect of Harmonic analysis. Substantial research in both domains has notably emphasized the spectral theory aspects, reflecting a significant level of attention [1–15].

In this study, the main achievement is the creation of a generator system that makes a group of easy-to-understand representations for the real eigenbasis of the discrete Fourier transform. The application of this approach leads to the emergence of the discrete analogue of the Hermite functions.

The discrete Fourier matrix, denoted as $\mathbf{F}_n$ with an indication of its size being n, is defined as follows.

$$\mathbf{F}_n = \frac{1}{\sqrt{n}}\Big(w^{kl}\Big)_{k,l=0}^{n-1} \qquad w = \exp\frac{-2\pi i}{n}$$

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Table 1: The row/column scaling factors are given by given by $a_i = \frac{1}{2}$ for $k = 0$ and 1 else; $b_i = \frac{1}{2}$ for $i = n - 1$ and 1 else.

Type	$(s_{kl})_{k,l=0}^{n-1}$	Type	$(c_{kl})_{k,l=0}^{n-1}$
$\mathbf{S}_{(1)}$	$\sqrt{\frac{2}{n+1}} \sin \frac{(k+1)(l+1)\pi}{n+1}$	$\mathbf{C}_{(1)}$	$\sqrt{\frac{2}{n-1}} a_l b_l \cos \frac{kl\pi}{n-1}$
$\mathbf{S}_{(5)}$	$\sqrt{\frac{2}{n+\frac{1}{2}}} \sin \frac{2(k+1)(l+1)\pi}{2n+1}$	$\mathbf{C}_{(5)}$	$\frac{2}{\sqrt{2n-1}} a_l \cos \frac{2kl\pi}{2n-1}$

It is demonstrated that any eigenbasis of $\mathbf{F}$ can be obtained through straightforward modifications of the eigenbases of the discrete trigonometric transforms (DTT), as outlined in Table 1. When the size of $\mathbf{F}$ is even, it is illustrated that any eigenbasis of $\mathbf{C}_{(1)}$ leads to an eigenspace of $\mathbf{F}$ corresponding to the eigenvalues ± 1. Furthermore, it is shown that any eigenbasis of $\mathbf{S}_{(1)}$ transforms directly into an eigenspace of $\mathbf{F}$ associated with the eigenvalues $\pm i$. The process remains analogous for odd values of n, utilizing the matrices $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$.

In Section 2, it is shown that the DTTs listed in Table 1 collectively belong to a specific class of orthogonal matrices. This shared characteristic facilitates a unified approach for the eigenbasis applicable to each member. The introduction of this larger class of matrices introduces a prototype for the DTTs, requiring a unified method for their eigenbases.

One notable benefit of this research is its clear exploration of the *minimal Hermite-type basis* that forms an orthonormal basis of the centered discrete Fourier transform as outlined in [2, 3]. It is a discrete analogue of the Hermite functions ψ_n.

$$\psi_n(x) = (-1)^n (\sqrt{\pi} 2^n n!)^{\frac{-1}{2}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}$$

It is a well-established fact that Hermite functions constitute a complete orthonormal basis of $L^2(\mathbb{R})$, comprising eigenfunctions of the continuous Fourier transform $\mathfrak{F}$ [16, 17].

$$\mathfrak{F}(g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ity} g(y) dy$$

1.1. Motivation

This section, deeply motivated by [1], illustrates how the DTTs defined in Table 1 establish a generation system for producing the eigenbasis of the DFT. It is a well-known that

$$\mathbf{F}_n^4 = \mathbf{I}_n$$

where $\mathbf{I}_n$ represents the identity matrix of dimension n. This leads to the conclusion that $\mathbf{F}_n$ possesses precisely four eigenvalues, namely $\{\pm 1, \pm i\}$, when $n \geq 4$. Consequently, for sufficiently large matrix dimensions, an infinite array of bases, consisting of the eigenvectors exist.

By utilizing the symmetry existing in each row of $\mathbf{F}_n$, as demonstrated in the appendix Section 5.2, along with the converters Γ and Λ provided in Table 2, one can readily determine

Table 2: $\mathbf{J}_m$ is the anti-diagonal matrix of size m

Size	$\Gamma \in M_{n,m+2}(\mathbb{R})$	$\Lambda \in M_{n,m}$
$n = 2m+2$	$1 \oplus \begin{pmatrix} \mathbf{I}_m & 0 \\ 0 & 1 \\ \mathbf{J}_m & 0 \end{pmatrix}$	$\begin{pmatrix} 0_{1\times m} \\ -\mathbf{I}_m \\ 0_{1\times m} \\ \mathbf{J}_m \end{pmatrix}$
$n = 2m+1$	$1 \oplus \begin{pmatrix} \mathbf{I}_m \\ \mathbf{J}_m \end{pmatrix}$	$0 \oplus \begin{pmatrix} -\mathbf{I}_m \\ \mathbf{J}_m \end{pmatrix}$

if $n = 2m+2$:

$$\mathbf{F}_n \circ \Gamma = \begin{pmatrix} \mathbf{C}_{(1)} \\ \mathbf{J}'_m \, \mathbf{C}_{(1)} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_n \circ \Lambda = i \begin{pmatrix} 0 \\ \mathbf{S}_{(1)} \\ 0 \\ -\mathbf{J}_m \, \mathbf{S}_{(1)} \end{pmatrix} \tag{1.1}$$

where $\mathbf{J}'_m = (0 \;\; \mathbf{J}_m \;\; 0)$ is a $m \times (m+2)$ matrix. It yeilds that,

$$\begin{cases} \mathbf{F}_n \Big(\Gamma(\mathbf{w})\Big) = \pm\Gamma(\mathbf{w}) \Leftrightarrow \mathbf{C}_{(1)}(\mathbf{w}) = \pm\,\mathbf{w} \\ \mathbf{F}_n \Big(\Lambda(\mathbf{w})\Big) = \mp i\Lambda(\mathbf{w}) \Leftrightarrow \mathbf{S}_{(1)}(\mathbf{w}) = \pm\,\mathbf{w} \end{cases}$$

Similarly when $n = 2m+1$,

$$\mathbf{F}_n \circ \Gamma = \begin{pmatrix} \mathbf{C}_{(5)} \\ \mathbf{J}''_m \, \mathbf{C}_{(5)} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_n \circ \Lambda = 0 \oplus i \begin{pmatrix} \mathbf{S}_{(5)} \\ -\mathbf{J}_m \, \mathbf{S}_{(5)} \end{pmatrix} \tag{1.2}$$

where $\mathbf{J}'' = [0 \mid \mathbf{J}_m]$ is in $M_{m,m+1}$. It yields that,

$$\begin{cases} \mathbf{F}_n \Big(\Gamma(\mathbf{w})\Big) = \pm\Gamma(\mathbf{w}) \Leftrightarrow \mathbf{C}_{(5)}(\mathbf{w}) = \pm\,\mathbf{w} \\ \mathbf{F}_n \Big(\Lambda(\mathbf{w})\Big) = \mp i\Lambda(\mathbf{w}) \Leftrightarrow \mathbf{S}_{(5)}(\mathbf{w}) = \pm\,\mathbf{w} \end{cases}$$

For a given matrix A, let us denote $\mathrm{Eig}_A(\lambda)$ as the eigenspace of A corresponding to the eigenvalue λ. In light of this, the aforementioned observations yield the following crucial insights.

Theorem 1.1. *Suppose $n = 2m+2$ and consider the convertors Γ and Λ given in the Table 2. Consider the transforms $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$ of size $m+2$ and m, respectively.*

1. Γ *maps* $\mathrm{Eig}_{\mathbf{C}_{(1)}}(\pm 1)$ *onto* $\mathrm{Eig}_{\mathbf{F}_n}(\pm 1)$.
2. Λ *maps* $\mathrm{Eig}_{\mathbf{S}_{(1)}}(\pm 1)$ *onto* $\mathrm{Eig}_{\mathbf{F}_n}(\mp i)$.

Theorem 1.2. *Suppose $n = 2m+1$. Consider the transforms $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$ of size $m+1$ and m, respectively.*

1. Γ *maps* $\mathrm{Eig}_{\mathbf{C}_{(5)}}(\pm 1)$ *onto* $\mathrm{Eig}_{\mathbf{F}_n}(\pm 1)$.
2. Λ *maps* $\mathrm{Eig}_{\mathbf{S}_{(5)}}(\pm 1)$*)* *onto* $\mathrm{Eig}_{\mathbf{F}_n}(\mp i)$.

Consequently, in order to achieve the eigenbasis of $\mathbf{F}_n$, it suffices to concentrate on the eigenspaces of the previously mentioned discrete sine and cosine transforms. It will be comprehensively discussed in the next section.

2. DTT-type matrices

We denote $\mathcal{R}_n$ as the set of all $n \times n$ real symmetric matrices A satisfying $A^2 = \mathbf{I}_n$. It is equivalent to write,

$$\begin{cases} A(A+\mathbf{I}_n) = (A+\mathbf{I}_n) \\ A(A-\mathbf{I}_n) = -(A-\mathbf{I}_n) \end{cases} \tag{2.1}$$

The first statement asserts that the non-zero columns of $A+\mathbf{I}_n$ are eigenvectors of A corresponding to $+1$, while the second statement confirms that the non-zero columns of $A-\mathbf{I}_n$ are eigenvectors corresponding to -1. Furthermore,

$$\mathrm{rank}(A+\mathbf{I}_n) + \mathrm{rank}(A-\mathbf{I}_n) = n \tag{2.2}$$

As a result, the $2n$ columns derived from the matrices $A \pm \mathbf{I}_n$ can jointly serve as a basis for $\mathbb{R}^n$, encompassing the eigenvectors of matrix A.

Definition 2.1. For a matrix A belonging to $M_n(\mathbb{R})$, we define A as having the Chebyshev property if its bottom-left submatrix, which is an $[\frac{n}{2}] \times [\frac{n}{2}]$ matrix, is nonsingular.

Example 2.2. As given in the appendix, Theorem 5.2, the symmetric discrete trigonometric transforms of size n, outlined in Table 1, are part of $\mathcal{R}_n$ and exhibit the satisfying Chebyshev property.

Theorem 2.3. *Let* $n = 2m+1$ *and* A *be in* $\mathcal{R}_n$ *and assume that it satisfies the Chebyshev property. There are only two possibilities* $-1, 1$ *for the trace of* A*. If* $\mathrm{Tr}\, A = 1$*,*

1st Approach *The first* $(m+1)$ *columns of* $A+\mathbf{I}_n$ *constitute a maximally linearly independent set associated with the eigenvalue* $+1$*. Furthermore, the first* m *columns of the matrix* $A-\mathbf{I}_n$ *comprise a maximally linearly independent set associated with the eigenvalue* -1*.*

2ed Approach *The last* $(m+1)$ *columns of the matrix* $A+\mathbf{I}_n$ *represent a maximally linearly independent set associated to the eigenvalue* $+1$*. Additionally, the last* m *columns of the matrix* $A-\mathbf{I}_n$ *form a maximally linearly independent set associated with the eigenvalue* -1*.*

If $\mathrm{Tr}\, A = -1$*, then*

1st Approach *The first* $(m+1)$*-columns of* $A-\mathbf{I}_n$ *form a maximal linearly independent set corresponded to the eigenvalue* -1*. Moreover, the first* m*-columns of* $A+\mathbf{I}_n$ *form a maximal linearly independent set corresponded to the eigenvalue* $+1$*.*

2ed Approach *The last $(m+1)$-columns of $A - \mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue -1. Moreover, the last m-columns of $A + \mathbf{I}_n$ form a maximal linearly independent set corresponded to the eigenvalue $+1$.*

Proof. Our emphasis lies in the 1st approach, given that the second approach is comparably derived using the symmetry property.

The subsequent notations will aid in comprehending the progression of the proof.

$$\begin{cases} A_{u;l,m} \text{ is the upper-left submatrix of } A \text{ with size } m. \\ A_{b;l,m} \text{ is the bottom-left submatrix of } A \text{ with size } m \\ A_{b;r,m} \text{ is the bottom-right submatrix of } A \text{ with size } m \\ \mathrm{Eigen}(A) \text{ is the set of eigenvalues of } A \\ \mathcal{E}_\lambda(A) \;\text{ is the eigenspace of } A \text{ corresponding to the eigenvalue } \lambda \end{cases}$$

Let H_+ (or H_-) denote the rectangular $n \times m$ matrix, with its columns consisting of the last m columns of $A + \mathbf{I}_n$ (or $A - \mathbf{I}_n$). By directly leveraging the orthogonality of matrix A, we obtain the following equalities.

$$H_+^t H_+ = 2\big(A_{b;r,m} + \mathbf{I}_m\big) \tag{2.3}$$

$$H_-^t H_- = 2\big(A_{b;r,m} - \mathbf{I}_m\big) \tag{2.4}$$

Firstly, we ascertain that both matrices $H_+^t H_+$ and $H_-^t H_-$ are indeed invertible. The initial m rows of H_+ (or H_-) precisely match the submatrix $A_{b;l,m}$, a direct result of the symmetry inherent in matrix A. This deduction indicates that both H_+ (or H_-) possess a left inverse. As a result, the square matrices $H_+^t H_+$ (or $H_-^t H_-$) achieve invertibility. Through the use of formulas (2.3) and (2.4), it can be inferred that neither -1 nor 1 exist within the eigenvalue spectrum of $A_{b;r,(\frac{n-1}{2})}$.

Through the application of Lemma 2.6, we can articulate

$$\mathrm{Eign}\big(A_{u;l,(\frac{n+1}{2})}\big) = \{-\lambda : \lambda \in \mathrm{Eign}\big(A_{b;r,(\frac{n-1}{2})}\big)\} \bigcup \{x\}$$

for some $x \in \mathbb{R}$. According to the Cauchy Interlacing theorem, it is affirmed that x serves as an eigenvalue of matrix A. As a result, x can only take on the values of either -1 or $+1$. Moreover,

$$\mathrm{Tr}\, A = \mathrm{Tr}\, A_{b;r,(\frac{n-1}{2})} + \mathrm{Tr}\, A_{u;l,(\frac{n+1}{2})} = x$$

In the event that $\mathrm{Tr}\, A = 1$, it follows that -1 is no longer an eigenvalue of $A_{u;l,(\frac{n+1}{2})}$, which is equivalent to stating that the initial $(\frac{n+1}{2})$ columns of $A + \mathbf{I}_n$ constitute a linearly independent set. Likewise, the intended result can be achieved in the case where $\mathrm{Tr}\, A = -1$. □

A brief review of the proof provided above instantly reveals the following result.

Theorem 2.4. *Let $n = 2m$ and A be in $\mathcal{R}_n$ and assume that it satisfies the Chebyshev property. Then $\mathrm{Tr}\, A = 0$. Furthermore,*

1st Approach *The first m columns of the matrix* $A + \mathbf{I}_n$ *(or* $A - \mathbf{I}_n$*) form a maximally linearly independent set associated with the eigenvalue* $+1$ *(or* -1*).*

2ed Approach *The last m columns of the matrix* $A+\mathbf{I}_n$ *(or* $A-\mathbf{I}_n$*) represent a maximally linearly independent set associated with the eigenvalue* $+1$ *(or* -1*).*

Corollary 2.5. *Let* A *be in* $\mathcal{R}_n$ *and assume that it satisfies the Chebyshev property.*

1. *Suppose that* $n = 2m$. *Both eigenvalues* $+1$ *and* -1 *share the same multiplicity, and this multiplicity is equal to* m.
2. *Consider the case where* $n = 2m + 1$. *In this situation if* $\mathrm{Tr}(A) = 1$, *the multiplicity of of* $+1$ *is* $m + 1$ *while the multiplicity of* -1 *is* m.

Consider matrix M in $M_n(\mathbb{R})$. We define a pair of principal submatrices B and C of matrix M as complementary if there exist rectangular matrices X and Y such that

$$M = \begin{pmatrix} B & X \\ Y & C \end{pmatrix}$$

The following technical lemma provides the necessary underpinning for Theorem (2.3).

Lemma 2.6. *Let* M *be in* $M_n(\mathbb{R})$ *with* $M^2 = \mathbf{I}_n$ *and suppose that* B *and* C *are a pair of complementary principal submatrices of* M. *Let* λ *be a scaler with* $|\lambda| \neq 1$.

i) λ *is an eigenvalue of matrix* B *if and only if* $-\lambda$ *is an eigenvalue of matrix* C.

ii) The geometrical multiplicity of the eigenvalue λ *for matrix* B *is equal to the geometrical multiplicity of the eigenvalue* $-\lambda$ *for matrix* C.

Proof. Suppose that $B \in M_k(\mathbb{R})$ and put

$$M = \begin{pmatrix} B & X \\ Y & C \end{pmatrix}$$

Through the utilization of the orthogonality property of matrix M, the following identities can be obtained.

$$\begin{cases} XY = \mathbf{I}_k - B^2 \\ CY = -YB. \end{cases} \tag{2.5}$$

Suppose λ is an eigenvalue of B with $|\lambda| \neq 1$. The first equality confirms that Y never vanishes on any eigenvector of B corresponding to λ. Utilizing this observation and applying the second identity, we deduce that $-\lambda$ is an eigenvalue of C. Furthermore, Y constitutes a one-to-one linear mapping from $\mathcal{E}_\lambda(B)$ to $\mathcal{E}_{-\lambda}(C)$. To extend this argument, relying on the equalities in (2.6), we can establish that X serves as a one-to-one linear mapping from $\mathcal{E}_{-\lambda}(C)$ to $\mathcal{E}_\lambda(B)$.

$$\begin{cases} YX = \mathbf{I}_k - C^2, \\ BX = -XC. \end{cases} \tag{2.6}$$

□

To reinforce the stated theorem 1.1 and 1.2, we will put forth multiple specific cases concerning the eigenbases of discrete sine and cosine transforms in the forthcoming discussion.

Example 2.7. Theorems 2.3 and 2.4 present two different approaches for obtaining the eigenbases of the transforms $\mathbf{C}_{(1)}, \mathbf{S}_{(1)}, \mathbf{C}_{(5)}$, and $\mathbf{S}_{(5)}$.

Example 2.8. Let's employ $\mathbf{T}$ to denote each of the transforms $\mathbf{C}_{(1)}, \mathbf{S}_{(1)}, \mathbf{C}_{(5)}$, and $\mathbf{S}_{(5)}$. When $\mathbf{T}$ has a size of $m = 2k+1$, the union of the first $k+1$ columns of $\mathbf{T}+\mathbf{I}_m$ and the last column of $\mathbf{T}+\mathbf{I}_m$ constitutes a basis of $\mathrm{Eig}_{\mathbf{T}}(+1)$.

Example 2.9. Suppose that $m = 2k$ is even. The union of even columns from the matrices $\mathbf{C}_{(1)} \pm \mathbf{I}_m$ (resp. $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) constitutes a basis consisting of eigenvectors of the matrix $\mathbf{C}_{(1)}$ (resp. $\mathbf{S}_{(1)}$). To verify this point, it is directly verified that for both matrices $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$, the entries (a_{kl}) exhibit column vector symmetry as follows.

$$\begin{cases} a_{n+1-j,l} = a_{j,l} & l \text{ is odd} \\ a_{n+1-j,l} = -a_{j,l} & l \text{ is even} \end{cases} \tag{2.7}$$

Through the application of this vector column symmetry condition, we deduce that,

$$\mathbf{I}^t_{\text{even}}(\mathbf{C}_{(1)} \pm \mathbf{J}_m)(\mathbf{C}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{even}} = \mathbf{I}_k$$

$$\mathbf{I}^t_{\text{even}}(\mathbf{S}_{(1)} \pm \mathbf{J}_m)(\mathbf{S}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{even}} = \mathbf{I}_k$$

Here, $\mathbf{I}_{\text{even}}$ is the rectangular matrix whose columns consist of the even columns of $\mathbf{I}_m$.

Similarly, the union of odd columns from the matrices $\mathbf{C}_{(1)} \pm \mathbf{I}_m$ (resp. $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) constitutes a basis consisting of eigenvectors of the matrix $\mathbf{C}_{(1)}$ (resp. $\mathbf{S}_{(1)}$).

$$\mathbf{I}^t_{\text{odd}}(\mathbf{C}_{(1)} \mp \mathbf{J}_m)(\mathbf{C}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{odd}} = \mathbf{I}_k$$

$$\mathbf{I}^t_{\text{odd}}(\mathbf{S}_{(1)} \mp \mathbf{J}_m)(\mathbf{S}_{(1)} \pm \mathbf{I}_m)\mathbf{I}_{\text{odd}} = \mathbf{I}_k$$

Here, $\mathbf{I}_{\text{odd}}$ is the rectangular matrix whose columns consist of the odd columns of $\mathbf{I}_m$.

Remark 2.10. Using the equation $\mathbf{F}_n = 1 \oplus \mathbf{J}_{n-1}$, the transformations $\mathbf{P}_{+1}, \mathbf{P}_{-1}, \mathbf{P}_{+i}$, and $\mathbf{P}_{-i}$ serve as projections onto eigenspaces of $\mathbf{F}$ corresponding to the eigenvalues $+1, -1, +i$, and $-i$ [9].

$$\begin{cases} \mathbf{P}_{+1} = \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big) + \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n \\ \mathbf{P}_{-1} = \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big) - \frac{1}{4}\Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n \\ \mathbf{P}_{+i} = \frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big) - i\frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n \\ \mathbf{P}_{-i} = \frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big) + i\frac{1}{4}\Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big)\mathbf{F}_n \end{cases}$$

Assume that $n = 2m+2$. Let $\mathbf{C}_{(1)}$ and $\mathbf{S}_{(1)}$ have sizes $m+2$ and m, respectively. Through a direct computation, it can be verified that the conversion of the l-th column of $\mathbf{C}_{(1)} \pm \mathbf{I}_{m+2}$ (or $\mathbf{S}_{(1)} \pm \mathbf{I}_m$) using Γ (or Λ) coincides with the l-th column $\mathbf{P}_{\pm 1}$ (or $\mathbf{P}_{\pm i}$).

In a similar vein, when $n = 2m+1$ and the sizes of $\mathbf{C}_{(5)}$ and $\mathbf{S}_{(5)}$ are $m+1$, the conversion of the l-th column of $\mathbf{C}_{(5)} \pm \mathbf{I}_{m+1}$ (or $\mathbf{S}_{(5)} \pm \mathbf{I}_{m+1}$) using Γ (or Λ) corresponds to the l-th column of $\mathbf{P}_{\pm 1}$ (or $\mathbf{P}_{\pm i}$).

Table 3: The eigenvalue multiplicity of $\mathbf{F}_n$.

n	$\lambda=+1$	$\lambda=-1$	$\lambda=-i$	$\lambda=+i$
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

The combination of Theorems 1.1, 1.2, 2.3 and 2.4 leads to the following conclusion.

Corollary 2.11. *Let m_λ represent the multiplicity of the eigenvalue λ of $\mathbf{F}$. The formula for m_λ is provided in Table 3. For any eigenvalue λ of $\mathbf{F}$, let's consider the submatrix $\mathbf{P}'_\lambda$ of $\mathbf{P}_\lambda$ introduced as follows.*

1. *In the case where $\lambda=\pm1$, $\mathbf{P}'_\lambda$ is obtained by selecting the first m_λ columns from the projection $\mathbf{P}_\lambda$.*
2. *If λ takes values of either $\pm i$, the submatrix $\mathbf{P}'_\lambda$ is composed of the columns within the projection matrix $\mathbf{P}_\lambda$ ranging from 2 to $m_\lambda+1$.*

The column vectors within $\mathbf{P}'_\lambda$ serve as an eigenbasis of $\mathbf{F}$ associated with the eigenvalue λ.

3. Exploring Hermite-Type Eigenbases in DFT

Here, our goal is to present a second approach for deriving a discrete analogue to the Hermite functions, as previously discussed in [2, 3]. Before delving into this, it is necessary to clarify the term *discrete analogue*. Setting the benchmark, the following characterization theorem 3.1 precisely describes that the Hermite functions stand as the unique orthonormal eigenfunctions of the continuous Fourier transform, satisfying the uncertainty principle.

Theorem 3.1. *Assume that the functions $f_n:\mathbb{R}\to\mathbb{R}$ satisfy the following conditions.*

Orthonormal eigenbasis *$\{f_n\}$ forms an orthonormal basis of $L^2(\mathbb{R})$ consisting of eigenfunctions of the continuous Fourier transform with $\mathfrak{F}(f_n)=(-i)^n f_n$.*

Hardy's uncertainty principle *For every $n>0$,*

$$\lim_{|x|\to\infty} x^{-(n+1)} e^{\frac{x^2}{2}} f_n(x)=0$$

For each positive integer n, it holds true that either f_n equals ψ_n or f_n is equal to $-\psi_n$.

This observation may pave the way for an apt counterpart in the discrete analogue of Hermite functions, especially when uncovering a discrete rendition of Hardy's uncertainty principle. This particular aspect is skillfully addressed in [2], and for the sake of completeness, we revisit it in the following discussion.

Definition 3.2. The width($\mathbf{v}$) of a real discrete signal $\mathbf{v}$ with a periodicity of n is defined the minimum value of k that satisfies the following condition:

$$\begin{cases} \mathbf{v}[-k] \neq 0 \text{ or } \mathbf{v}[k] \neq 0 \\ \mathbf{v}[j] = 0 \text{ for all } |j| > k \end{cases}$$

The following characterization theorem, previously substantiated in [2], introduces a discrete alternative to Theorem 3.1.

Theorem 3.3. *Let us consider the centered discrete Fourier transform $\mathcal{F}$ of size n defined by*

$$\mathcal{F} = \left(w^{kl}\right) \qquad \textit{where} \quad [-\frac{n}{2}] + 1 \leq k, l \leq [\frac{n}{2}]$$

There exists a unique sequence of vectors $\{\mathbf{V}_k\}_{k=0}^{n-1}$ that adheres to the subsequent set of conditions.

Orthonoraml eigenbasis *The sequence $\{\mathbf{V}_k\}$ constitutes a orthonormal basis of $\mathbb{R}^n$ that is composed of eigenvectors of $\mathcal{F}$. Specifically, for each k in $0, \ldots, n-1$, the eigenvector $\mathbf{V}_k$ corresponds to the eigenvalue $(-i)^k$ of $\mathcal{F}$.*

Uncertainty principle *width*$(\mathbf{V}_k) \leq [\frac{n+k+2}{4}]$ *where* $k = 0, \ldots, n-1$.

The orthonormal basis outlined in Theorem 3.3 is denoted as the minimal Hermite-type basis of $\mathbb{R}^n$. With the approach outlined in this study, a new proof will be presented through an innovative method. As a second approach, we illustrate that the minimal Hermite-type basis can be smoothly acquired by leveraging the eigenspaces obtained from the columns of the submatrices $\mathbf{P}'_\lambda$ outlined in Corollary 2.11. The proof is structured in two parts. We recall that λ is applied to any eigenvalues, specifically those in the set $\{\pm 1, \pm i\}$ of $\mathbf{F}$, each with the multiplicity of m_λ.

Part1. Let us consider the forward shift of size n.

$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ \mathbf{I}_{n-1} & 0 \end{pmatrix}$$

Let $q = [\frac{n}{2}]$. We have,

$$\mathbf{F}_n = \mathbf{S}^{-q}\, \mathcal{F}_n\, \mathbf{S}^q \tag{3.1}$$

Then $\mathbf{S}^q$ corresponds a linear ismorophism from $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ onto $\mathrm{Eig}_{\mathcal{F}_n}(\lambda)$.

Part2. By employing the Gram-Schmidt algorithm on a minor modification of the columns within the submatrices $\mathbf{P}'_\lambda$, a unique orthonormal basis, consisting of eigenvectors of $\mathbf{F}$ is established. The application of the forward shift operator to transform these eigenvectors precisely generates the minimal Hermite-type basis. To address minor modification, we need to use a submatrix of $\mathbf{P}'_\lambda$ defined as follows. Consider the matrix B_λ, termed as a sparser matrix, as the submatrix with dimensions $m_\lambda \times m_\lambda$ of $\mathbf{P}'_\lambda$, as outlined below:

Suppose the size is $n = 2m + 1$. The rows of matrix $\mathbf{P}'_\lambda$ range from $m+2-m_\lambda$ to $m+1$ form the matrix B_λ.

Suppose the size is $n = 2m + 2$. For $\lambda = \pm 1$, the rows of $\mathbf{P}'_\lambda$ range from $m + 3 - m_\lambda$ to $m+2$ introduce B_λ. In the case of $\lambda = \pm i$, the rows of span from $m+2-m_\lambda$ to $m+1$ of $\mathbf{P}'_\lambda$.

As validated in the appendix, the symmetry property of $\mathbf{F}$ causes the matrix B_λ to exhibit the Chebyshev property, signifying its non-singularity.

Proof of Theorem 3.3. The nonsingularity of the sparser matrix B_λ ensures the existence of vectors $\mathbf{x}_1^{(\lambda)}, \ldots, \mathbf{x}_{m_\lambda}^{(\lambda)}$ in $\mathbb{R}^{m_\lambda}$ with

$$\mathbf{x}_k^{(\lambda)} = B_\lambda^{-1} \mathbf{e}_k^{(\lambda)} \qquad k = 1, \ldots, m_\lambda \tag{3.2}$$

where $\mathbf{e}_k^{(\lambda)}$ is the standard vector in $\mathbb{R}^{m_\lambda}$ such that all its components are zero, with the exception of a single component in the k-th position, which has a value of 1. We define,

$$\mathbf{w}_k^{(\lambda)} = \mathbf{P}'_\lambda \mathbf{x}_k^{(\lambda)} \qquad k = 1, \ldots, m_\lambda \tag{3.3}$$

The set of vectors $\{\mathbf{w}_k^{(\lambda)}\}$ form a basis of $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$. When n is odd,

$$\mathbf{w}_k^{(\lambda)} = \begin{bmatrix} \clubsuit \\ \mathbf{e}_k^{(\lambda)} \\ \alpha\, \mathbf{J}\left(\mathbf{e}_k^{(\lambda)}\right) \\ \clubsuit \end{bmatrix} \quad \text{where} \quad \alpha = \begin{cases} 1 & \lambda = \pm 1 \\ -1 & \lambda = \pm i \end{cases}$$

Here, the symbol $\clubsuit$ denotes the part filled by some scalars.

We proceed with the process when n is odd. In the case of even n, the same outcome is obtained, with only a slight modification in the formula of $\mathbf{w}_k^{(\lambda)}$.

By the Gram-Schmidt process, we can transform the set $\{\mathbf{w}_k^{(\lambda)}\}$ into $\{\mathbf{z}_k^{(\lambda)}\}$, constructing an orthogonal basis for $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ in a way that collectively forms the columns of the following matrix $\mathbf{Z}^{(\lambda)}$.

$$\mathbf{Z}^{(\lambda)} = [\mathbf{z}_1^{(\lambda)} | \mathbf{z}_2^{(\lambda)} | \cdots | \mathbf{z}_{m_\lambda}^{(\lambda)}] = \begin{bmatrix} \clubsuit \\ \mathbf{U} \\ \alpha\, \mathbf{J}_{m_\lambda} \mathbf{U} \\ \clubsuit \end{bmatrix} \tag{3.4}$$

where $\mathbf{U}$ is $m_\lambda \times m_\lambda$ upper-triangular matrix whose diagonal entries are all non-zero. The non-singularity of matrix $\mathbf{U}$ ensures the uniqueness of $\{\mathbf{z}_k^{(\lambda)}\}$ as the orthogonal basis for $\mathrm{Eig}_{\mathbf{F}_n}(\lambda)$ that satisfies the specified pattern in reference to (3.4).

The set of vectors $\{\mathbf{S}^q\,\mathbf{z}_k^{(\lambda)}\}_{k=1}^{m_\lambda}$ serves as an orthonormal basis for $\mathrm{Eig}_{\mathcal{F}}(\lambda)$. Furthermore,

$$width\left(\mathbf{S}^q\,\mathbf{z}_k^{(\lambda)}\right) \leq [\frac{n}{2}] - m_\lambda + k$$

Now, it is time to introduce the desired orthonormal basis $\mathbf{V}_k$. Initially, we organize all vectors $\{\mathbf{S}^q\,\mathbf{z}_k^{(\lambda)}\}$ in four vertical columns. The first column contains vectors associated with $\lambda = 1$, the second with $\lambda = -1$, the third with $\lambda = -i$, and the fourth with $\lambda = i$. Moving in each row upwards to cover all $\mathbf{S}^q\,\mathbf{z}_k^{(\lambda)}$, the vectors $\mathbf{V}_k$ are defined as follows.

$$\begin{array}{cccc}
\lambda = 1 & \lambda = -1 & \lambda = -i & \lambda = i \\
\mathbf{V}_0 = \mathbf{z}_1^{(1)} & \mathbf{V}_1 = \mathbf{z}_1^{(-1)} & \mathbf{V}_2 = \mathbf{z}_1^{(-i)} & \mathbf{V}_3 = \mathbf{z}_1^{(i)} \\
\mathbf{V}_4 = \mathbf{z}_2^{(1)} & \mathbf{V}_5 = \mathbf{z}_2^{(-1)} & \mathbf{V}_6 = \mathbf{z}_2^{(-i)} & \mathbf{V}_7 = \mathbf{z}_2^{(i)} \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

4. Figures

Depicted in Figure 1 is the distribution of the minimal Hermite basis $\{\mathbf{V}_k\}$ for the size $n = 43$. Part (a) of Figure 1 illustrates their arrangement based on width, with shorter vectors appearing before longer ones. Moreover, part (b) of Figure 1 validates that each vector $\mathbf{V}_k$ contains some other zeros within its middle components, underscoring the sparsity of the minimal Hermite-type basis.

While no sample of the Hermite eigen-functions $\{\psi_k\}$ precisely forms a basis of eigen vectors for DFT, the minimal Hermite-type basis approximately addresses this issue. To elaborate further, let's define,

$$\mathbf{\Psi}_j(k) = (\sqrt{\frac{2\pi}{n}})^{\frac{1}{k}}\psi_j(\sqrt{\frac{2\pi}{n}}k)$$

It is proved in [2]

$$\| \mathbf{V}_k - \mathbf{\Psi}_k \| = O(n^{-1+\epsilon}) \qquad \text{as} \;\; n \to \infty \tag{4.1}$$

Figure 2 presents a comparison between eight types of minimal Hermite-type vectors $\{\mathbf{V}_k\}$ and their corresponding Hermite eigenfunctions $\{\psi_k\}$. An observable pattern suggests that vectors $\mathbf{V_k}$ with shorter widths offer closer approximations to their corresponding Hermite functions. Nevertheless, this observation lacks analytical proof.

The uniqueness of the minimal Hermite-type basis highlights an important aspect. This basis stands out as the most effective sparse vectors for approximating the Hermite eigenfunctions. To accurately determine the approximation rate of the Hermite-type basis compared to that mentioned in (4.1), it is necessary to replace the uncertainty principle cited in Theorem 3.3 with another property.

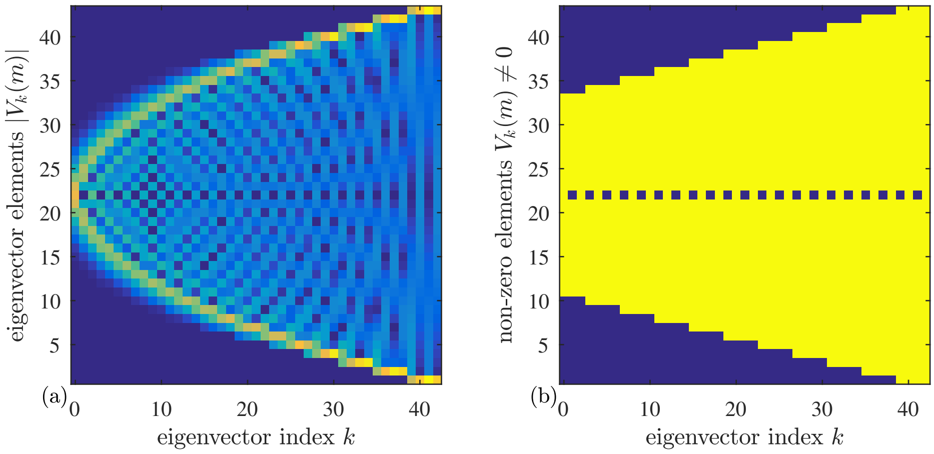

Figure 1: Orthonormal eigenvectors $\mathbf{V}_k$ for $n = 43$: (a) absolute value, (b) non-zero elements.

5. Appendix

5.1. Chebyshev property

In this section, points are presented to reinforce the Chebyshev property of DTTs in this study.

Definition 5.1. A sequence of functions $\{\phi_1, \ldots, \phi_n\}$ is called a Chebyshev system on an interval J, if any linear combination $\phi = \sum_{l=1}^{n} a_l \phi_l$ has at most $n-1$ distinct zeros J. It is equivalent to say that for every strictly increasing sequence $t_1, \ldots, t_n$ in J, the $n \times n$ matrix $\Phi = \Big(\phi_l(t_k)\Big)$ is invertible.

$$\Phi = \begin{pmatrix} \phi_1(t_1) & \cdots & \phi_n(t_1) \\ \vdots & \ddots & \vdots \\ \phi_1(t_n) & \cdots & \phi_n(t_n) \end{pmatrix} \tag{5.1}$$

The following theorem is demonstrated in [1].

Theorem 5.2. *the sequence* $\{\cos lx\}_{l=0}^{n}$ *(and* $\{\sin lx\}_{l=1}^{n}$*) over the interval* $[0, \pi]$ *(over* $(0, \pi)$*) forms a Chebyshev system.*

Applying the same approach leads us to obtain the following Chebyshev systems.

1. Both sequences $\{\cos(2l-1)x\}_{l=1}^{n}$ and $\{\cos(2l)x\}_{l=1}^{n}$ are Chebyshev systems on the closed interval $[0, \frac{\pi}{2}]$.
2. The sequences $\{\sin(2l-1)x\}_{l=1}^{n}$ and $\{\sin(2l)x\}_{l=1}^{n}$ are Chebyshev systems on the open interval $(0, \frac{\pi}{2})$.

These facts directly imply the Chebyshev property of the discrete sine and cosine matrices in Table (1).

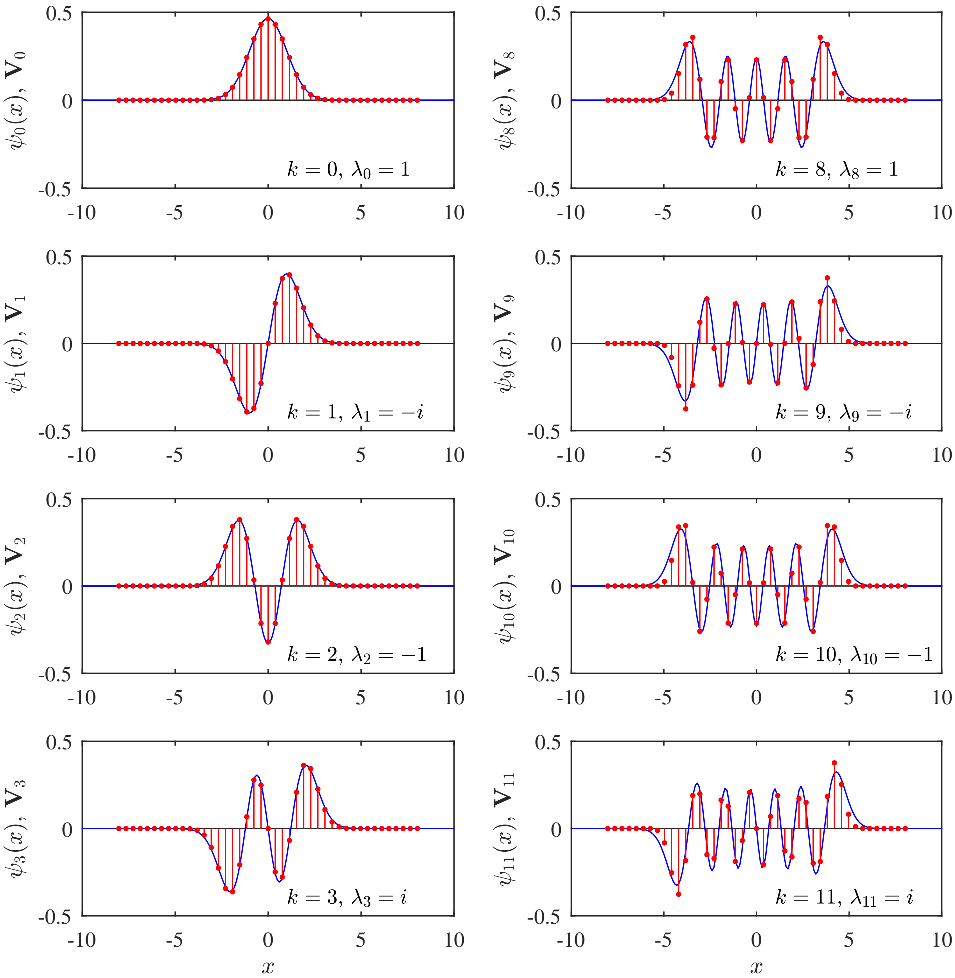

Figure 2: Eigenvectors $\mathbf{V}_k$ for $k = 0, 1, 2, 3, 8, 9, 10, 11$ and $n = 43$ along with corresponding continuous Hermite functions $\psi_k(x)$.

The presented trigonometric equations assure us that all these discrete trigonometric transformations are orthogonal matrices. For details, we refer to [18].

$$\sum_{k=0}^{n-1} \cos(kt) = \frac{\cos\left(\frac{n-1}{2}t\right)\sin\left(\frac{n}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \tag{5.2}$$

5.2. Symmetry in $\mathbf{F}_n$

Recalling $w = \exp -\frac{2\pi i}{m}$, we proceed to define,

$$\mathbf{G}_m = \left(w^{kl}\right)_{k,l=1}^{m} \qquad \text{and} \qquad \mathbf{G}_m^H = \left(w^{-kl}\right)_{k,l=1}^{m} \tag{5.3}$$

We have then,

$$n = 2m+1: \qquad \mathbf{F}_n = \begin{bmatrix} 1 & 1_{1\times m} & 1_{1\times m} \\ 1_{m\times 1} & \mathbf{G}_m & \mathbf{G}_m^H \mathbf{J}_m \\ 1_{m\times 1} & \mathbf{J}_m \mathbf{G}_m^H & \mathbf{J}_m \mathbf{G}_m \mathbf{J}_m \end{bmatrix}$$

$$n = 2m+2: \qquad \mathbf{F}_n = \begin{bmatrix} 1 & 1_{1\times m} & * & 1_{1\times m} \\ 1_{m\times 1} & \mathbf{G}_m & \vdots & \mathbf{G}_m^H \mathbf{J}_m \\ 1_{m\times 1} & \mathbf{J}_m \mathbf{G}_m^H & * & \mathbf{J}_m \mathbf{G}_m \mathbf{J}_m \end{bmatrix}$$

The column where the entries are denoted by $*$ is just:

$$[1 \ -1 \ 1 \ -1 \ \cdots \ (-1)^{n-1}]^t$$

It directly leads to the following formulas.

$$\begin{cases} \Big(\mathbf{I}_n + (1 \oplus \mathbf{J}_{n-1})\Big)\,\mathbf{F}_n = \frac{2}{\sqrt{n}}\left(\cos\frac{2kl\pi}{n}\right)_{k,l=0}^{n-1} \\ \Big(\mathbf{I}_n - (1 \oplus \mathbf{J}_{n-1})\Big)\,\mathbf{F}_n = -i\frac{2}{\sqrt{n}}\left(\sin\frac{2kl\pi}{n}\right)_{k,l=0}^{n-1} \end{cases} \tag{5.4}$$

Let $\mathbf{E} = (a_{kl})$ represent the $n \times n$ matrix, where all elements are zero except for $a_{1,n} = 1/4$. By taking into account equations (5.4), we obtain the formula for the sparser matrices B_λ, which is discussed in the proof of Theorem 3.3.

$$B_{-1} = \begin{cases} \frac{-1}{2\sqrt{n}}\left(\cos\frac{2(k+m_{-1}+1)l\pi}{n}\right)_{k,l=0}^{m_{-1}-1} & n \equiv_4 0 \ \text{ or } \ n \equiv_4 1 \\ \frac{-1}{2\sqrt{n}}\left(\cos\frac{2(k+m_{-1})l\pi}{n}\right)_{k,l=0}^{m_{-1}-1} & n \equiv_4 2 \ \text{ or } \ n \equiv_4 3 \end{cases}$$

$$B_{1} = \begin{cases} \frac{1}{2\sqrt{n}}\left(\cos\frac{2(k+m_{\lambda}-1)l\pi}{n}\right)_{k,l=0}^{m_{1}-1} + \mathbf{E} & n \equiv_4 0 \ \text{ or } \ n \equiv_4 1 \\ \frac{1}{2\sqrt{n}}\left(\cos\frac{2(k+m_{\lambda})l\pi}{n}\right)_{k,l=0}^{m_{1}-1} & n \equiv_4 2 \ \text{ or } \ n \equiv_4 3 \end{cases}$$

$$B_{i} = \begin{cases} \frac{-1}{2\sqrt{n}}\left(\sin\frac{2(k+m_{i}+2)(l+1)\pi}{n}\right)_{k,l=0}^{m_{i}-1} & n \equiv_4 0 \ \text{ or } \ n \equiv_4 3 \\ \frac{-1}{2\sqrt{n}}\left(\sin\frac{2(k+m_{i}+1)(l+1)\pi}{n}\right)_{k,l=0}^{m_{i}-1} & n \equiv_4 1 \ \text{ or } \ n \equiv_4 2 \end{cases}$$

$$B_{-i} = \begin{cases} \frac{1}{2\sqrt{n}}\left(\sin\frac{2(k+m_{-i})(l+1)\pi}{n}\right)_{k,l=0}^{m_{-i}-1} + \mathbf{E} & n \equiv_4 0 \text{ or } n \equiv_4 3 \\ \frac{1}{2\sqrt{n}}\left(\sin\frac{2(k+m_{-i}+1)(l+1)\pi}{n}\right)_{k,l=0}^{m_{-i}-1} & n \equiv_4 1 \text{ or } n \equiv_4 2 \end{cases}$$

Regarding these sparser matrices B_λ, apart from those perturbed by the matrix $\mathbf{E}$, the remaining matrices simply exhibit the Chebyshev property. To tackle the Chebyshev property of these elements, we utilize the following lemma, which is an interesting point in and of itself.

Lemma 5.3. *Suppose that $A = (a_{kl})$ is an $n \times n$ non-singular matrix. For a given scalar x, let us define $A_x = (b_{kl})$ with,*

$$b_{kl} = \begin{cases} x & k = 1, l = n \\ a_{kl} & \textit{otherwise} \end{cases}$$

There exists at most one scalar value x, that causes A_x to become singular.

Proof. Scalars s and t exist such that $\det(A) = sa_{1n} + t$. The fact that matrix A is non-singular implies that at least one of the scalars s or t must be non-zero. When $s = 0$, then $\det A_x = t$ meaning that scalars x make A_x non-singular. While $s \neq 0$, the matrix is singular if and only if $x = -\frac{t}{s}$. □

Now, let's shift our focus to matrix B_1 when $n \equiv_4 0$, as the remaining matrices are treated in a similar manner.

$$B_1 = \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k + m_\lambda - 1)l\pi}{n} \right)_{k,l=0}^{m_1-1} + \mathbf{E}$$

Let us define.

$$\tilde{B_1} = B_1 - 2\mathbf{E} = \frac{1}{2\sqrt{n}} \left(\cos \frac{2(k + m_\lambda - 1)l\pi}{n} \right)_{k,l=0}^{m_1-1} - \mathbf{E}$$

It's important to note that the matrix $\tilde{B_1}$ is definitely singular, as if it were nonsingular, the multiplicity of the eigenvalue -1 of $\mathbf{F}_n$ would become m_1, which is impossible. Utilizing Lemma 5.3, we can therefore conclude that B_1 is nonsingular.

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