Refining Directed Graphs for Spectral Analysis through Strategic Topological Modifications

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***Abstract*—In graph signal processing, the absence of a well-defined graph Fourier transform complicates spectral analysis on directed graphs. To address this, we propose a technique that subtly modifies the graph's topology, making minimal changes to facilitate spectral analysis.**

Our approach starts by adding edges to eliminate sources and sinks, followed by identifying connections that impede the non-singularity and diagonalizability of the adjacency matrix. We then make minor edge weight adjustments to achieve diagonalizability, ensuring efficient spectral analysis of graph signals defined on directed graph.

***Index Terms*—Graph Fourier transform, Directed graph, Eigendecomposition, Jordan form**

I. Introduction

Graphs are increasingly being explored and utilized in signal processing due to their versatility and effectiveness in representing complex structures. This trend arises from the fact that almost all real-world phenomena can be represented as graphs, making them more comprehensible not only to experts but also to individuals not specializing in data (or signal) processing.

Signal processing on graphs has significant applications beyond scientific and theoretical research. It can be used in the analysis of various networks, including the internet, traffic, telecommunications, social networks etc. [3], [9]. By leveraging graph-based signal processing, it becomes possible to uncover intricate patterns and relationships within these networks, leading to more insightful analysis and practical applications.

The Fourier transform is a fundamental tool for analyzing and understanding signal behavior by decomposing signals into their constituent frequency components. This work is motivated by extending the discrete Fourier transform (DFT) to general directed graphs, which allows for the analysis of signals defined on graph structures. Several approaches are explored to achieve this extension, including techniques such as zero-padding [7], [8], Möbius transformations [4], [5], various usage methods of Laplacian matrix [1], [2], [6].

In this work, an innovative solution is proposed, comprising two primary elements: (i) augmenting the directed graph (digraph) with additional edges to establish connections to both sink and source vertices, thereby producing a modified digraph that is free of sinks and sources, and (ii) identifying a minimal subset of existing edges such that minor modifications to their weights make the adjacency matrix diagonalizable. This transformation facilitates the application of the Fourier transform for spectral analysis of signals defined on directed graph.

The paper is organized as follows. Section 2 provides a comprehensive overview of the Fourier transform, including both the classical version and its adaptation to graph structures. This section will also detail the differences in applying the Fourier transform to undirected versus directed graphs. Section 3 offers a detailed explanation of the proposed solution to the problem, including the mathematical foundation and practical implementation considerations. Section 4 presents the results and examples, demonstrating the effectiveness of the proposed method through simple numerical example. Finally, Section 5 concludes the paper, summarizing the findings and discussing potential future directions for research in this area.

II. Preliminaries

The Fourier Transform is a mathematical tool that converts signals from the time or spatial domain into the frequency domain, enabling a more detailed analysis and manipulation of signals in the frequency domain. It is widely utilized today as one of the most common tools for signal analysis in engineering, physics, and signal processing. One specific form of the Fourier Transform is the Discrete Fourier Transform. The DFT is designed to transform a sequence of discrete

This work is partially supported by the Montenegrin Ministry of Education, Science and Innovations, project grant: DPG "Data Processing on Graphs", and by the Montenegrin Academy of Sciences and Arts.

samples from the time domain into the frequency domain, making it a valuable tool for analyzing the frequency content of discrete signals. Mathematically, the DFT is defined for a sequence of N samples and is expressed as follows:

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-i\frac{2\pi}{N}kn} \tag{1}$$

- $X(k)$ is the DFT coefficient for frequency index k
- $x[n]$ is the time-domain sample at index n
- $e^{-i\frac{2\pi}{N}kn}$ is a complex exponential representing the basis functions of the DFT
- N is the total number of samples
- k ranges from 0 to $N-1$, representing the frequency bins.

A. Fourier Transformation on Graphs

A directed graph is defined as an ordered pair (G, E), where:

- G is a set of vertices (nodes),
- E is a set of directed edges where each edge is an ordered pair of vertices (u, v) indicating a directed edge from vertex u to vertex v.

The adjacency matrix A associated with a directed graph (G, E) is a square matrix defined as follows:

- Let $G = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices, where $n = |G|$.
- The adjacency matrix A is an $n \times n$ matrix where the entry A_{ij} is defined as:

$$A_{ij} = \begin{cases} 1 & \text{if there is a directed edge from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The Jordan Normal Form (JNF) of the adjacency matrix A is a canonical form of A obtained through similarity transformation. Specifically, the JNF of A is a block diagonal matrix J that can be expressed as:

$$A = VJV^{-1} \tag{2}$$

where V is an invertible matrix whose columns are just the generalized eigenvectors of A. The matrix J consists of Jordan blocks J_i, which are square matrices associated with each eigenvalue λ_i of A. Each Jordan block J_i has the following structure:

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}$$

where λ_i is an eigenvalue of A.

In the realm of graph signal processing, the matrix $F = V^{-1}$ is referred to as the graph Fourier transform matrix. By leveraging the following equations, which pertain to graph Fourier analysis and synthesis, the fundamental objectives of graph signal processing can be effectively accomplished on a theoretical level:

Graph Fourier analysis: This process involves transforming a graph signal from its original vertex domain representation to the graph frequency domain using the graph Fourier transform, F. Mathematically, it can be expressed as $x \in \mathbb{R}^N$, a signal on the graph, transformed into its graph frequency representation, $\hat{x}$, using the equation:

$$\hat{x} = Fx \tag{3}$$

Graph Fourier synthesis: This process entails reconstructing a graph signal in the vertex domain from its graph frequency domain representation by applying the inverse graph Fourier transform, F^{-1}. Mathematically, the synthesis of x from its frequency representation, $\hat{x}$, can be described as:

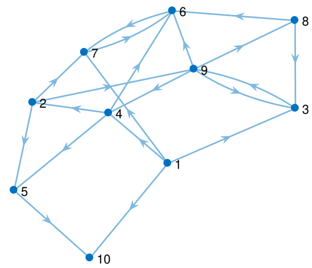
$$x = F^{-1}\hat{x} \tag{4}$$

The presence of a higher number of Jordan blocks of size greater than 1 in the matrix J has significant implications for the stability of the graph Fourier transform, F. As the number of these larger Jordan blocks increases, the graph Fourier transform becomes increasingly unstable. This instability arises from the sensitivity of the transform to small perturbations in the graph structure or signal values. Consequently, the analysis and synthesis of graph signals can become more challenging and susceptible to numerical errors, affecting the accuracy and reliability of the results. Therefore, understanding and managing the stability of the graph Fourier transform, particularly in cases where the number of large Jordan blocks is substantial, is crucial for effective graph signal processing. This may involve developing novel techniques or adapting existing methods to mitigate instability and ensure accurate, robust results for various applications in network science, machine learning, and other fields that rely on the analysis of signals on graphs.

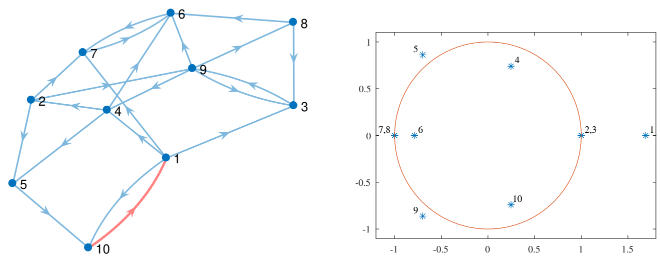
In the context of directed graphs, the cycle directed graph serves as the domain for signals in digital signal processing. Within this framework, the shift backward matrix is equivalent to the adjacency matrix of the cycle directed graph. The graph Fourier transform, as it appears in the Jordan Normal Form, is essentially equivalent to the discrete Fourier transform.

III. Minimal Changes for Eliminating Jordan Blocks

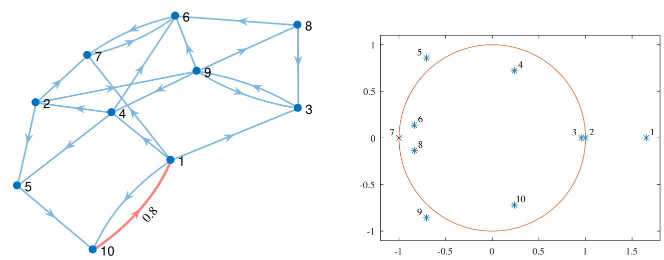
The presence of Jordan blocks in the Jordan decomposition of the adjacency matrix leads to non-diagonalizability, which can cause numerical instability in the graph Fourier transform. A critical question is: what minimal refinements to the entries of the adjacency matrix can effectively eliminate these Jordan blocks? Addressing this question involves identifying modifications to the topology of the directed graph that result in an adjacency matrix with no Jordan blocks. The optimal outcome of such refinements is a modified directed graph that supports a more stable and reliable graph Fourier transform, thereby potentially enhancing the efficiency and accuracy of graph signal processing.



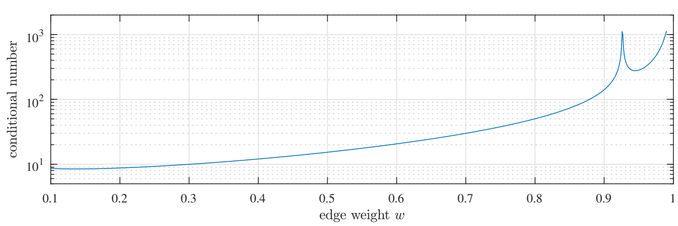
(a) Original graph



(b) Corrected graph - added edge is marked with thick red line



(c) Corrected graph - added weighted edge



(d) Coditional number of GFT matrix

Fig. 1. Original graph (a) has one sink and one source. Connecting sink and source with an edge (b) ensures non-singularity, but not diagonalizability of the adjacency matrix. Adding weighted edge (c) diagonalizability is achieved. Numerical stability of GFT is presented as conditional number of GFT matrix dependence on weight of the added edge (d).

Jordan Blocks Associated with the Eigenvalue 0. The number of Jordan blocks associated with the eigenvalue 0, known as the geometric multiplicity of 0, is directly related to the nullity index of the adjacency matrix, which is the dimension of its null space or kernel.

Step1. In a directed graph, sinks are indicated by zero columns in the adjacency matrix, while sources are indicated by zero rows. Both features are crucial for determining singularity. To address the elimination of Jordan blocks in these specific cases, it is necessary to introduce new connections by adding additional edges to existing sources and sinks. In a directed graph with N vertices, there are $N-1$ potential options for adding a new edge to connect to a source or sink, while avoiding loops.

If the nullity index is addressed solely by considering sinks and sources, leveraging non-unit weights on the new edge connections may resolve diagonalizability challenges. Otherwise, the procedure outlined in Step 2 should be followed.

Step2. Assume that there is neither a source nor a sink in the directed graph. In this case, to identify vertices that contribute to the formation of Jordan blocks associated with the eigenvalue 0, it is sufficient to examine which rows (or columns) of the adjacency matrix can be expressed as linear combinations of the others. The maximal set of rows (or columns) whose exclusion does not affect the rank of the adjacency matrix identifies a set of vertices where minor adjustments to the weights of the connecting edges can eliminate all corresponding Jordan blocks. The sets of such vertices for rows and columns are typically non-unique.

The modifications proposed in Steps 1 and 2 will definitively eliminate all Jordan blocks associated with the eigenvalue 0. The nullity index of the adjacency matrix determines the number of required operations to achieve this goal.

Empirical evidence suggests that after completing these two steps, the appearance of additional non-trivial Jordan blocks (associated with non-zero eigenvalues) is exceptionally rare in most real-world datasets. In the uncommon instances where Jordan blocks associated with non-zero eigenvalues persist, it is recommended to adjust edge weights by making slight modifications to the weights of few edges added in Step 1 or Step 2 until the desired outcome is achieved. In this way we minimize changes in the graph structure needed to achieve diagonalizability of the adjacency matrix. This process may also be applied to other edges (from the original graph) if needed.

IV. Example

In the following section, a specific example is provided to illustrate the proposed solution. Figure 1 (a) represents a graph with 10 vertices and 20 edges that has been constructed to demonstrate the application and effectiveness of the approach, whose adjacency matrix formula is given as follows.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There is one source and one sink in the graph, as indicated by the zero column (the first one) and zero row (the last one) in matrix A. This implies that A is singular. Moreover, the presence of a non-trivial Jordan block associated with the eigenvalue 0 in the Jordan Normal Form confirms that A is not diagonalizable.

To address this issue, as outlined in the approach, the only remedy is to add a new edge from the sink to another vertex.

In Figure 1 (b), a new edge, highlighted in red, is proposed. This modification addresses the singularity of the matrix; however, it is not sufficient to achieve diagonalizability. Additional adjustments, such as adding more edges or altering the weight of the new edge, may be necessary to fully ensure diagonalizability. As as illustrated in Figure 1 (c) altering the weight of the new edge can address the challenge. Indeed, considering any weight less than 1 leads to diagonalizability. To determine an appropriate weight, we need to logically track the stability of the corresponding GFT arising from each of these weights. The conditional number of the GFT for a range of weights on this new edge is analyzed in Figure 1 (d). As observed, smaller weights correspond to higher numerical stability.

In signal processing, eigenvalues located on the unit circle generally indicate a stable system, characterized by the absence of exponential growth or decay. The distribution of the eigenvalues at each step is also enclosed in Figure 1. This configuration implies that the signals undergo transformation or evolution while maintaining a consistent amplitude, thereby preserving energy over time without amplification or attenuation. Such stability is crucial for the effective functioning of filters, as it ensures accurate signal processing and prevents distortion. This property is particularly significant in applications such as digital filters and control systems, where maintaining stable oscillations and consistent signal characteristics is essential for optimal performance.

Conclusion

This research successfully addresses the challenge of applying Fourier transform techniques to directed graphs, an essential tool for analyzing the frequency components of graph signals. The proposed approach involves augmenting the original directed graph with additional edges of weights less than one, ensuring that the resulting adjacency matrix becomes invertible and diagonalizable. This modification facilitates the definition and application of the Fourier transform on the augmented graph.

Our findings indicate that the introduction of low-weight edges results in a modified graph that remains similar to the original in terms of signal processing characteristics. This ensures that the augmented graph maintains the structural integrity required for accurate signal analysis.

These results have significant implications for the fields of social networks, communication networks, and other domains where complex interactions are represented as graphs. By enabling the use of Fourier transform on directed graphs, our method enhances the ability to detect patterns and anomalies, thereby offering deeper insights into the structure and dynamics of these systems. The practical applicability and theoretical significance of our approach highlight its potential as a robust tool for graph signal processing.

Further work The approach outlined here raises the following challenging question for future work: In real datasets, we often encounter directed graphs with multiple sinks and sources, where introducing new edge connections may not resolve singularity. Moreover, various sets of new connections and different weight adjustments theoretically address singularity and fix diagonalizability issues. Identifying the most appropriate option poses a significant challenge, both theoretically and numerically.

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